Math 118: PDE

HW 6 Solutions

4.1.1

- A violin string is modeled exactly by the problem (1), (2), and (3).
- Its general solution is given as a Fourier expansion in (9).
- We find that the frequencies (or note produced by the violin string) are

$$\frac{n\pi\sqrt{T}}{l\sqrt{\rho}} \quad \text{for } n = 1, 2, 3, ...,$$

  which depends on length of string l and tension force T.

- The frequencies double when l is decreased by half.
- The frequencies are also proportinal to $\sqrt{T}$. Thus, the frequencies increase, i.e, the note rises, when T increases, i.e., when the string is tightened.

4.1.2

- The problem is given by

$$\begin{aligned} \text{DE} &: u_t = k u_{xx} \quad (0 < x < l, 0 < t < \infty) \\ \text{BC} &: u(0,t) = u(l,t) = 0 \\ \text{IC} &: u(x,0) = 1. \end{aligned}$$

- We have

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l}$$

  is the solution provided that

$$1 = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

- Assuming the infinite series expansion

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right),$$

we obtain

$$A_n = \begin{cases} \frac{4}{n\pi} & \text{if n is odd} \\ 0 & \text{otherwise.} \end{cases}$$

- The formula for the temperature $u(x,t)$ at later times is

$$u(x,t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n} e^{-(n\pi/l)^2 kt} \sin\frac{n\pi x}{l}$$

4.1.3

- Plugging the form $u(x,t) = X(x)T(t)$ into the Schrodinger's equation, we get

$$X(x)T'(t) = iX''(x)T(t),$$

or dividing by iXT,

$$\frac{T'}{iT} = \frac{X''}{X} = -\lambda.$$

- The quantity λ must be a positive constant. (This will be shown at the end of the section.) We get a pair of separate ODE:

$$X'' + \lambda X = 0, \quad T' + i\lambda T = 0.$$

- Using (8), we see that

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l} \quad (n = 1, 2, 3, \dots)$$

are distinct solutions that satisfy the Dirichlet conditions.

- Solving for T, we obtain

$$Tn(t) = e^{-i\lambda_n t}.$$

- The general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} e^{-i(n\pi/l)^2 t}.$$

4.1.6

- Plugging the form $u(x,t) = X(x)T(t)$ into the equation, we get

$$tX(x)T'(t) = X''(x)T(t) + 2X(x)T(t),$$

or

$$tX(x)T'(t) - 2X(x)T(t) = X''(x)T(t).$$

- Dividing by XT, we obtain

$$\frac{tT'}{T} - 2 = \frac{X''}{X} = -\lambda.$$

- Using (8) with $l = \pi$, we see that

$$\lambda_n = n^2, \quad X_n(x) = \sin(nx) \quad (n = 1, 2, 3, \dots)$$

are distinct solutions that satisfy the given boundary conditions.

- Solving for T, we get

$$T_n(t) = t^{2-\lambda_n}.$$

- The general solution is

$$u(x,t) = \sum_1^\infty A_n \sin(nx)\, t^{2-n^2},$$

which satisfies the initial condition $u(x,0) = 0$ for all A_n. Thus, the solution is not unique.

5.1.1

- Let $x = \pi/4$, we have

$$\sin nx = \begin{cases} \sqrt{2}/2 & \text{if } n = 8k+1 \text{ or } 8k+3, \\ -\sqrt{2}/2 & \text{if } n = 8k-1 \text{ or } 8k-3, \\ 0 & \text{otherwise.} \end{cases},$$

where k is an integer.

- The expansion $1 = \sum_{n \text{ odd}} (4/n\pi) \sin nx$ can be rewritten as

$$1 = \frac{4}{\pi}\frac{\sqrt{2}}{2}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots\right).$$

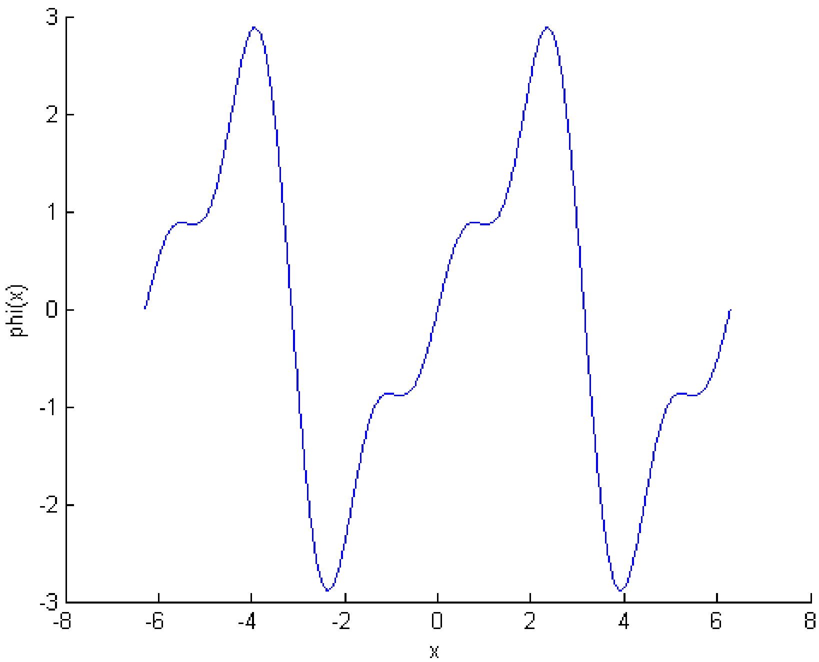
- The sum $1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots$ is thus given by $\pi\sqrt{2}/4$.
- Read the hint.

5.1.2(a)

- Its Fourier since series has coefficients

$$\begin{aligned}
A_m &= 2\int_0^1 x^2 \sin m\pi x \, dx \\
&= -\frac{2x^2}{m\pi}\cos m\pi x\Big|_0^1 + \frac{4}{m\pi}\int_0^1 x\cos m\pi x \, dx \\
&= -\frac{2x^2}{m\pi}\cos m\pi x\Big|_0^1 + \frac{4}{m\pi}\left(\frac{x}{m\pi}\sin m\pi x + \frac{1}{m^2\pi^2}\cos m\pi x\right)\Big|_0^1 \\
&= -\frac{2}{m\pi}\cos m\pi + \frac{4}{m^3\pi^3}(\cos m\pi - 1) \\
&= -\frac{2}{m\pi}(-1)^m + \frac{4}{m^3\pi^3}((-1)^m - 1) \\
&= \begin{cases} \frac{2}{m\pi} & \text{for n even} \\ \frac{2}{m\pi} - \frac{4}{m^3\pi^3} & \text{for n odd} \end{cases}.
\end{aligned}$$

5.1.3(a)



5.1.8

- The problem is modeled by

$$\begin{aligned}\text{DE} &: u_t = k u_{xx} \quad (0 < x < 1, 0 < t < \infty)\\ \text{BC} &: u(0,t) = 0, \quad u(1,t) = 0\\ \text{IC} &: u(x,0) = \phi(x),\end{aligned}$$

  where

$$\phi(x) = \begin{cases} 5x/2 & \text{for } 0 < x < 2/3 \\ 3 - 2x & \text{for } 2/3 < x < 1. \end{cases}$$

- The equilibrium solution of this problem is given by $U(x) = x$.

- Let $v(x,t) = u(x,t) - x$, then $v(x,t)$ is the solution of the problem

$$\begin{aligned}\text{DE} &: v_t = k v_{xx} \quad (0 < x < 1, 0 < t < \infty)\\ \text{BC} &: v(0,t) = v(1,t) = 0\\ \text{IC} &: v(x,0) = \phi(x) - x = \psi(x),\end{aligned}$$

where

$$\psi(x) = \begin{cases} 3x/2 & \text{for } 0 < x < 2/3 \\ 3 - 3x & \text{for } 2/3 < x < 1. \end{cases}$$

- We have

$$v(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l}$$

is the solution provided that

$$\psi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

- Using the Fourier sine series for 1 and x in the interval $(0, l)$ on Page 104-105, we can simply get

$$A_n = \begin{cases} (-1)^{m+1} \frac{3l}{m\pi} & \text{for } 0 < x < 2/3 \\ \frac{6}{m\pi}[1 - (-1)^m - (-1)^{m+1} l] & \text{for } 2/3 < x < 1. \end{cases}$$