Math 118: PDE

HW 7 Solutions

4.1.4

- Using the separated solution in the PDE, we get

$$F\ddot{G} = c^2 F'' G - rF\dot{G}.$$

Separation of variables gives

$$\frac{\ddot{G} + r\dot{G}}{c^2 G} = \frac{F''}{F} = -\lambda,$$

where λ is a separation constant.

- The eigenvalue problem for $F(x)$ is

$$F'' + \lambda F = 0,\ F(0) = 0, F(l) = 0.$$

The eigenvalues and eigenfunctions are

$$F_n(x) = \sin\left(\frac{n\pi x}{l}\right), \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, 3, \ldots.$$

- The ODE for $G(t)$ is

$$\ddot{G} + r\dot{G} + c^2\lambda G = 0.$$

Its characteristic equation is $w^2 + rw + c^2\lambda = 0$, which has solutions

$$w = \frac{-r \pm \sqrt{r^2 - 4c^2\lambda}}{2}.$$

Given that $0 < r < 2\pi c/l$, we have $r^2 < 4c^2\lambda$ for all $n \geq 1$. This means the roots are complex. Let $w_n = \alpha \pm i\beta_n$, where $\alpha = -\frac{r}{2}, \beta_n = \frac{\sqrt{4c^2\lambda_n - r^2}}{2}$. The solution is given by

$$G(t) = e^{\alpha t}\left(A_n \cos\beta_n t + B_n \sin\beta_n t\right).$$

- The separated solutions are therefore

$$u(x,t) = \sin\left(\frac{n\pi x}{l}\right) e^{\alpha t}\left(A_n \cos\beta_n t + B_n \sin\beta_n t\right).$$

- Taking a linear combination of the separated solutions, we get that the genral solution of the PDE and BCs is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) e^{\alpha t} \left(a_n \cos \beta_n t + b_n \sin \beta_n t\right).$$

- Imposition of the initial condition $u(x,0) = \phi(x)$ gives

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right),$$

  so a_n is the nth Fourier coefficient of $\phi(x)$,

$$a_n = \frac{2}{l} \int_0^l \phi(x) \sin\left(\frac{n\pi x}{l}\right)\, dx.$$

- Imposition of the initial condition $u_t(x,0) = \psi(x)$ gives

$$\psi(x) = \sum_{n=1}^{\infty} b_n \beta_n \sin\left(\frac{n\pi x}{l}\right),$$

  so $b_n \beta_n$ is the nth Fourier coefficient of $\psi(x)$, and

$$b_n = \frac{2}{l\beta_n} \int_0^l \psi(x) \sin\left(\frac{n\pi x}{l}\right)\, dx.$$

4.2.1

- Using the separated solution in the PDE, we get

$$F\dot{G} = kF''G.$$

  Separation of variables gives

$$\frac{\dot{G}}{kG} = \frac{F''}{F} = -\lambda,$$

  where λ is a separation constant.

- The eigenvalue problem for $F(x)$ is

$$F'' + \lambda F = 0,\ F(0) = 0, F'(l) = 0.$$

The eigenvalues and eigenfunctions are

$$F_n(x) = \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right), \quad \lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2, \quad n = 1, 2, 3, \ldots.$$

- The ODE for $G(t)$ is

$$\dot{G} + k\lambda G = 0.$$

Up to an arbitrary constant factor, the solution is

$$G(t) = e^{-k\lambda t}.$$

- The general solution is therefore

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{(n+\frac{1}{2})\pi x}{l}\right) e^{-k\left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2 t}.$$

4.2.4

- Using the separated solution in the PDE, we get

$$F\dot{G} = kF''G.$$

Separation of variables gives

$$\frac{\dot{G}}{kG} = \frac{F''}{F} = -\lambda,$$

where λ is a separation constant.

- The eigenvalue problem for $F(x)$ is

$$F'' + \lambda F = 0,\ F(-l) = F(l), F'(-l) = F'(l).$$

  - If $\lambda = 0$, then $F'' = 0$. The solution is $F(x) = Ax + B$. Imposition of the BCs gives $A = 0$. Therefore $F_0 = B$, where B is an arbitrary constant is a solution.

- If $\lambda > 0$, the solution is $F(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. Since the cos functions are even, imposition of the BCs implies $\sin(\sqrt{\lambda}l) = 0$, i.e., $\sqrt{\lambda}l = n\pi$ for integers n.
- Combining the two cases, the eigenvalues are $\lambda_n = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$.

The solution for $F(x)$ is therefore $F(x) = A\cos\frac{n\pi x}{l} + B\sin\frac{n\pi x}{l}$.

- The ODE for $G(t)$ is

$$\ldots G + k\lambda G = 0.$$

Up to an arbitrary constant factor, the solution is $G(t) = e^{-k\lambda t}$. If $\lambda = 0$, $G(t) = 1$.

- The general solution is therefore

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{l} + B_n\sin\frac{n\pi x}{l}\right)e^{-k\lambda t}.$$

5.1.4

- The Fourier cosine series is

$$|\sin x| = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx, \quad -\pi < x < pi,$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin x\,dx = \frac{4}{\pi}$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|\cdot\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cdot\cos nx\,dx,$$

because $|\sin x|$ is even. Since

$$\begin{aligned}\int_0^{\pi}\sin x\cdot\cos nx\,dx &= \frac{1}{n}\sin x\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos x\sin nx\,dx\\ &= -\frac{1}{n}\int_0^{\pi}\cos x\sin nx\,dx\\ &= \frac{1}{n^2}\cos x\cos nx\Big|_0^{\pi} + \frac{1}{n^2}\int_0^{\pi}\sin x\cos nx\,dx,\end{aligned}$$

we have

$$(1-\frac{1}{n^2})\int_0^{\pi} \sin x \cdot \cos nx \; dx = \frac{1}{n^2}\cos x \cos nx\Big|_0^{\pi}$$
$$= -\frac{1}{n^2}(\cos n\pi + 1)$$
$$= \begin{cases} -\frac{2}{n^2} \text{ if n is even} \\ 0 \text{ if n is odd} \end{cases}$$

It follows that

$$a_n = \begin{cases} \frac{4}{(1-n^2)\pi} \text{ if n is even} \\ 0 \text{ if n is odd} \end{cases}$$

- The Fourier cosine series of $|\sin x|$ is therefore

$$|\sin x| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n \text{even}} \frac{1}{1-n^2}\cos nx$$
$$= \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{1-4n^2}\cos 2nx.$$

- Plugging in $x = \pi$ into the previous Fourier cosine series, we get

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1},$$

and

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

- Plugging in $x = \frac{\pi}{2}$ into the found Fourier cosine series, we get

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1},$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}.$$

5.1.5

- The Fourier sine series of $\phi(x) = x$ is

$$\phi(x) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l}.$$

- Integrating term by term, we get

$$\frac{x^2}{2} = \frac{2l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{l} + C,$$

  where C is the constant of integration.

- The constant term in the cosine series is

$$a_0 = \frac{2}{l} \int_0^l \frac{x^2}{2}\, dx = \frac{l^2}{3}.$$

  Thus, $C = \frac{1}{2} a_0 = \frac{l^2}{6}$.

- Plugging $x = 0$ into the Fourier cosine series for $\frac{x^2}{2}$, we get

$$0 = \frac{2l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + \frac{l^2}{6},$$

  which gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

5.2.4

- Sine is odd; Cosine is even.
- The product of two even (or odd) functions is even; the product of an even function and an odd function is odd.
- The integral of an odd function over $(-l, l)$ is zero.

5.2.11

- The complex Fourier series of e^x is

$$e^x = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l},$$

where

$$\begin{aligned} c_n &= \frac{1}{2l}\int_{-l}^{l} e^x \cdot e^{-in\pi x/l}\, dx \\ &= \frac{1}{2l(1-in\pi/l)}\, e^{(1-in\pi/l)x}\Big|_{-l}^{l} \\ &= \frac{1}{2(l-in\pi)}\left[e^{l-in\pi} - e^{-l+in\pi}\right] \\ &= \frac{(-1)^n}{l-in\pi}\frac{e^l - e^{-l}}{2} \\ &= \frac{(-1)^n(l+in\pi)}{l^2+n^2\pi^2}\sinh l. \end{aligned}$$

- To get the real Fourier series of e^x, we first rewrite the complex series

$$\begin{aligned} e^x &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n(l+in\pi)}{l^2+n^2\pi^2}\sinh(l)e^{in\pi x/l} \\ &= \frac{\sinh l}{l} + \sum_{n=1}^{\infty}\frac{(-1)^n(l+in\pi)}{l^2+n^2\pi^2}\sinh(l)e^{in\pi x/l} + \sum_{n=1}^{\infty}\frac{(-1)^n(l-in\pi)}{l^2+n^2\pi^2}\sinh(l)e^{-in\pi x/l}. \end{aligned}$$

The real Fourier series of e^x is

$$e^x = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right),$$

where

$$a_0 = \frac{2\sinh l}{l},$$

$$a_n = \sinh l\left(\frac{(-1)^n(l+in\pi)}{l^2+n^2\pi^2} + \frac{(-1)^n(l-in\pi)}{l^2+n^2\pi^2}\right) = \frac{(-1)^n 2l\sinh l}{l^2+n^2\pi^2},$$

and

$$a_n = i\sinh l\left(\frac{(-1)^n(l+in\pi)}{l^2+n^2\pi^2} - \frac{(-1)^n(l-in\pi)}{l^2+n^2\pi^2}\right) = \frac{(-1)^{n+1}2n\pi\sinh l}{l^2+n^2\pi^2}.$$

- Note that $a_0 = 2c_0$, $a_n = c_n + c_{-n}, b_n = i(c_n - c_{-n})$ for $n = 1, 2, 3, \ldots$.

6.2.4

- Using the separated solution $u(x, y) = F(x)G(y)$ in the PDE, we get

$$F''G + G\ddot{G} = 0.$$

Separation of variables gives

$$\frac{F''}{F} = -\frac{\ddot{G}}{G} = \lambda.$$

- First, we solve the subproblem with BCs: $u(x, 0) = u(x, 1) = 0, u_x(0, y) = 0, u_x(1, y) = y^2$. The eigenvalue problem for $G(y)$ is

$$\ddot{G} + \lambda G = 0, \ G(0) = G(1) = 0.$$

The eigenvalues and eigenfunctions are

$$G_n(y) = \sin n\pi y, \quad \lambda_n = (n\pi)^2, \quad n = 1, 2, 3, \ldots.$$

The ODE for $F(x)$ is

$$F'' - \lambda_n F = 0,$$

with solutions

$$F_n(x) = a_n \cosh n\pi x + b_n \sinh n\pi x.$$

Imposiition of the BC $F'(0) = 0$ gives $B_n = 0$. Thus, $F(x) = a_n \cosh n\pi x$. The general solution is

$$u_1(x, y) = \sum_{n=1}^{\infty} a_n \cosh n\pi x \sin n\pi y,$$

where a_n must be chosen such that the solution satisfies the BC $u_x(1, y) = y^2$, i.e.,

$$a_n = \frac{2}{n\pi \sinh n\pi x} \int_0^1 y^2 \sin n\pi y \, dx.$$

- Next, we solve the subproblem with BCs: $u(x,0) = x, u(x,1) = 0, u_x(0,y) = u_x(1,y) = 0$. Using the same strategy, we find the general solution is

$$u_2(x,y) = \frac{a_0}{2}(y-1) + \sum_{n=1}^{\infty} \cos n\pi x \sinh n\pi(y-1),$$

where

$$a_0 = -1,\ a_n = -\frac{2}{\sinh n\pi}\int_0^1 x\cos n\pi x\ dx.$$

- Adding the solutions of the two subproblems, we get our final solution.

6.2.7

- Using the separated solution in the PDE, we get

$$F\ddot{G} + F''G = 0.$$

Separation of variables gives

$$-\frac{\ddot{G}}{G} = \frac{F''}{F} = -\lambda,$$

where λ is a separation constant.

- The eigenvalue problem for $F(x)$ is

$$F'' + \lambda F = 0,\ F(0) = 0, F(\pi) = 0.$$

The eigenvalues and eigenfunctions are

$$F_n(x) = \sin nx, \quad \lambda_n = n^2, \quad n = 1, 2, 3, \ldots.$$

- The ODE for $G(y)$ is

$$\ddot{G} - \lambda G = 0.$$

The solution is

$$G(y) = Ae^{\sqrt{\lambda}y} + Be^{-\sqrt{\lambda}y} = Ae^{ny} + Be^{-ny}.$$

Imposition of the BC $\lim_{y\to\infty} G(y) = 0$ gives $A = 0$, so

$$G(y) = Be^{-ny}.$$

- The general solution is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin nx e^{-ny}.$$

- Setting $y = 0$ in the series above, we require that

$$h(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin nx,$$

which gives

$$b_n = \frac{2}{\pi} \int_0^{\pi} h(x) \sin nx \; dx.$$

6.3.2

- The solution of the Laplace's equation on the disk is

$$u(r,\theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} r^n \left(A_n \cos n\theta + B_n \sin n\theta \right).$$

- Setting $r = a$ in the series above, we require that

$$1 + 3 \sin\theta = u(a,\theta) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} a^n \left(A_n \cos n\theta + B_n \sin n\theta \right),$$

which gives

$$A_0 = 2, \; B_1 = \frac{3}{a},$$

and all the other coefficients are zero.

- The solution of the BVP is therefore

$$u(r,\theta) = 1 + \frac{3r}{a} \sin\theta.$$

6.3.4

- Given
$$P(r,\theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos\theta + r^2},$$
we want to show
$$P_{rr} + \frac{1}{r}P_r + \frac{1}{r^2}P_{\theta\theta} = 0.$$

- We can do this by direct differentiation, and get
$$P_r = \frac{2a((s^2 + r^2)\cos\theta - 2ar)}{(a^2 - 2ar\cos\theta + r^2)^2},$$
$$P_{rr} = \frac{4a(a^3\cos(2\theta) - r(3a^2 + r^2)\cos\theta + 3ar^2)}{(a^2 - 2ar\cos\theta + r^2)^3},$$
and
$$P_{\theta\theta} = \frac{-2ar(a^2 - r^2)((a^2 + r^2)\cos\theta + ar(\cos(2\theta) - 3))}{(a^2 - 2ar\cos\theta + r^2)^3}.$$

- OR we can first write $P(r,\theta)$ in its series form
$$P(r,\theta) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n\theta,$$
and then integrate term by term.