Midterm 1: Sample solutions
Math 118A, Fall 2013

1. Say whether the following operators acting on functions $u(x, y)$ are linear or nonlinear. Justify your answers. (a) $Lu = u_{xx} + u_{yy} + 1$; (b) $Lu = yu_{xx} + u_{yy} + u$; (c) $Lu = uu_{xx} + u_{yy}$.

Solution.

- (a) Nonlinear because of the nonhomogeneous term 1. For example,

$$L(u + v) = Lu + Lv - 1 \neq Lu + Lv.$$

- (b) Linear. For every constant c and all functions u, v:

$$\begin{aligned} L(cu) &= y(cu)_{xx} + (cu)_{yy} + cu \\ &= c\left(yu_{xx} + u_{yy} + u\right) \\ &= cLu; \\ L(u + v) &= y(u + v)_{xx} + (u + v)_{yy} + u + v \\ &= yu_{xx} + u_{yy} + u + yv_{xx} + v_{yy} + v \\ &= Lu + Lv. \end{aligned}$$

- (c) Nonlinear because of the uu_{xx} term. For example, if $c \neq 1$,

$$L(cu) = c^2uu_{xx} + cu_{yy} = c\left(cuu_{xx} + u_{yy}\right) \neq cLu.$$

2. Solve the following IVP for the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = 0, \quad u_t(x,0) = \cos x.$$

Solution.

- D'Alembert's solution of the wave equation with initial data

  $$u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x)$$

  is

  $$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\, d\xi.$$

- Setting $\phi(x) = 0$ and $\psi(x) = \cos x$, we get

  $$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{x+ct} \cos\xi\, d\xi \\ &= \frac{1}{2c}\left[\sin(x+ct) - \sin(x-ct)\right]. \end{aligned}$$

3. Look for solutions of the heat equation

$$u_t = k u_{xx},$$

of the form

$$u(x,t) = f(x)e^{-a^2 t}$$

where $a > 0$ is a constant. Find the most general function $f(x)$ for which this is a solution. Give a physical explanation, in terms of heat flow, of why this solution decays exponentially in time.

Solution.

- For functions u of the form given in the question, we have

$$u_t = -a^2 f e^{-a^2 t}, \qquad u_{xx} = f'' e^{-a^2 t}.$$

  Thus, canceling the nonzero exponential factor, we see that u satisfies the heat equation if and only if $-a^2 f = k f''$, or

$$f'' + \frac{a^2}{k} f = 0.$$

- The general solution of this equation (we assume $k > 0$) is

$$f(x) = A \cos\left(\frac{ax}{\sqrt{k}}\right) + B \sin\left(\frac{ax}{\sqrt{k}}\right),$$

  where A, B are arbitrary constants. (The characteristic equation is $r^2 + a^2/k = 0$, with roots $r = \pm ia/\sqrt{k}$.)

- The solution decays because heat flows from hot spots where $f > 0$ to cold spots where $f < 0$. The faster u oscillates in space, the faster it decays in time. More quantitatively, if $u(x,t)$ has wavelength $\lambda = 2\pi\sqrt{k}/a$ in x, then it decays at at rate $e^{-\beta t}$ in time, where $\beta = 4\pi^2 k/\lambda^2$. The rate of decay is also larger for larger thermal diffusivities k.

4. For what values of the constants m, n does the PDE

$$u_t + uu_x + u_{xxx} = 0$$

have similarity solutions of the form

$$u(x,t) = \frac{1}{t^m} f\left(\frac{x}{t^n}\right)?$$

In that case, find an ODE for $f(z)$. (Don't try to solve it!)

Solution.

- For similarity solutions of the form given in the question, we have (by the chain rule)

$$\begin{aligned} u_t &= -\frac{m}{t^{m+1}} f - \frac{nx}{t^{m+n+1}} f', \\ u_x &= \frac{1}{t^{m+n}} f', \\ u_{xxx} &= \frac{1}{t^{m+3n}} f'''. \end{aligned}$$

- It follows that u is a solution of the PDE if

$$-\frac{m}{t^{m+1}} f - \frac{nx}{t^{m+n+1}} f' + \frac{1}{t^m} f \cdot \frac{1}{t^{m+n}} f' + \frac{1}{t^{m+3n}} f''' = 0.$$

  After multiplting this equation by t^{m+1}, we get

$$-mf - \frac{nx}{t^n} f' + \frac{1}{t^{m+n-1}} ff' + \frac{1}{t^{3n-1}} f''' = 0.$$

- We get a self-consistent solution of the PDE only if f does not depend on t except through $z = x/t^n$. This is the case only if the powers of t in front of the terms ff' and f''' are zero, meaning that $m + n - 1 = 0$ and $3n - 1 = 0$, or

$$m = \frac{2}{3}, \qquad n = \frac{1}{3}.$$

- In that case, $f(z)$ satisfies the ODE

$$f''' + ff' - \frac{1}{3} zf' - \frac{2}{3} f = 0.$$

5. Suppose that $u(x,t)$ is a solution of the initial value problem

$$u_t + cu_x + u = 0, \qquad u(x,0) = \phi(x).$$

such that u and its derivatives approach zero as $|x| \to \infty$. Show that

$$\int_{-\infty}^{\infty} u^2(x,t)\, dx = e^{-2t} \int_{-\infty}^{\infty} \phi^2(x)\, dx.$$

Solution.

- Multiplying the PDE by u and using $uu_t = (u^2/2)_t$ (and similarly for x-derivatives), we get

  $$\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{2}cu^2\right)_x + u^2 = 0.$$

  Integrating this equation with respect to x and using

  $$\int_{-\infty}^{\infty} (u^2)_t\, dx = \frac{d}{dt}\int_{-\infty}^{\infty} u^2\, dx, \qquad \int_{-\infty}^{\infty} (u^2)_x\, dx = u^2\Big|_{-\infty}^{\infty} = 0,$$

  we get that

  $$\frac{d}{dt}\int_{-\infty}^{\infty} u^2\, dx + 2\int_{-\infty}^{\infty} u^2\, dx = 0,$$

  or $y_t + 2y = 0$ where $y(t) = \int_{-\infty}^{\infty} u^2(x,t)\, dx$. Solving this ODE, we get

  $$\int_{-\infty}^{\infty} u^2\, dx = Ce^{-2t},$$

  and evaluating this expression at $t = 0$, we find that

  $$C = \int_{-\infty}^{\infty} \phi^2(x)\, dx,$$

  which proves the result.

- An alternative method is to solve the IVP exactly, which gives

  $$u(x,t) = e^{-t}\phi(x - ct),$$

  and then note that

  $$\int_{-\infty}^{\infty} u^2(x,t)\, dx = e^{-2t}\int_{-\infty}^{\infty} \phi^2(x-ct)\, dx = e^{-2t}\int_{-\infty}^{\infty} \phi^2(x)\, dx,$$

  where we change the integration variable from $x - ct$ to x in the last step. The first method doesn't depend on having an explicit solution.

6. Suppose that algae on a (one-dimensional) lake has population density $u(x,t)$. Assume that the algae grows at a rate proportional to its population density and diffuses from high-density to low density regions at a rate proportional to its population gradient u_x. Derive a PDE for $u(x,t)$.

Solution.

- For an arbitray interval $a \le x \le b$, we have

$$\begin{aligned}&\text{rate of change of algae population in } a \le x \le b\\&\qquad = (\text{flux of algae into } a \le x \le b)\\&\qquad + (\text{growth rate of population in } a \le x \le b)\end{aligned}$$

- Assume that $q = -ku_x$ is the flux of algae and $r = cu$ is the growth rate density. Then

$$\begin{aligned}\frac{d}{dt}\int_a^b u(x,t)\,dx &= q(a,t) - q(b,t) + \int_a^b r(x,t)\,dx\\&= -ku_x(a,t) + ku_x(b,t) + \int_a^b cu(x,t)\,dx.\end{aligned}$$

- Using the fundamental theorem of calculus to write

$$-ku_x(a,t) + ku_x(b,t) = \int_a^b ku_{xx}(x,t)\,dx,$$

  and combining the terms, we get that

$$\int_{-\infty}^{\infty} (u_t - ku_{xx} - cu)\ dx = 0.$$

- Since this equation holds for all $a < b$, the integrand $u_t - ku_{xx} - cu$ must be zero (assuming it's continuous), so the PDE is

$$u_t = ku_{xx} + cu.$$