Partial Differential Equations

Math 118A, Fall 2013

Midterm 1: Solutions

1. [15%] Is the PDE

$$yu_{xx} + u_{yy} = 0$$

linear or nonlinear? For what constants A, B, C is the function

$$u(x, y) = Ax^2y + Bxy^2 + Cy^4$$

a solution of the PDE?

Solution.

- The PDE is linear (and homogeneous with variable coefficients).
- For the given function,

  $$u_{xx} = 2Ay, \qquad u_{yy} = 2Bx + 12Cy^2,$$

  so

  $$yu_{xx} + u_{yy} = (2A + 12C)y^2 + 2Bx.$$

  This is zero if and only if $A = -6C$ and $B = 0$, so the most general such solution is

  $$u(x, y) = A\left(y^4 - 6x^2y\right).$$

2. [20%] Find the solution $u(x,t)$ of the following initial value problem for the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = \sin x, \quad u_t(x,0) = e^x.$$

Solution.

- Using d'Alembert's solution

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\, d\xi$$

  with $\phi(x) = \sin x$ and $\psi(x) = e^x$, we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} e^{\xi}\, d\xi \\ &= \frac{1}{2}\left[\sin(x+ct) + \sin(x-ct)\right] + \frac{1}{2c}\left[e^{x+ct} - e^{x+ct}\right]. \end{aligned}$$

3. [25%] (a) Give a particular solution of the PDE

$$u_t + 2u_x = x.$$

(b) What is the general solution of this PDE?
(c) Find the solution $u(x,t)$ of the initial value problem

$$u_t + 2u_x = x, \qquad u(x,0) = 0.$$

Solution.

- (a) We look for a particular solution $u = u_p(x)$ depending only on x. Then $2u_p'(x) = x$, so a particular solution is

$$u_p(x) = \frac{1}{4}x^2.$$

- (b) By linearity, the general solution has the form

$$u(x,t) = u_p(x) + v(x,t)$$

where v is a solution of the homogeneous equation $v_t + 2v_x = 0$. The general solution of this advection equation is $v = f(x - 2t)$, so

$$u(x,t) = \frac{1}{4}x^2 + f(x - 2t)$$

where $f(x)$ is an arbitrary function of integration.

- (c) Imposing the initial condition on the general solution, we get that

$$0 = \frac{1}{4}x^2 + f(x),$$

so $f(x) = -x^2/4$, and the solution is

$$u(x,t) = \frac{1}{4}x^2 - \frac{1}{4}(x - 2t)^2 = xt - t^2.$$

4. [20%] (a) Suppose that a, c, k are constants. Show that

$$u(x,t) = e^{at}v(x - ct, t)$$

is a solution of the PDE

$$u_t + cu_x = ku_{xx} + au$$

if and only if $v(x,t)$ is a solution of $v_t = kv_{xx}$.
(b) Give an expression for the solution of the initial value problem

$$u_t + cu_x = ku_{xx} + au, \qquad u(x,0) = \phi(x)$$

in terms of the source function

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}$$

of the heat equation.

Solution.

- (a) By the product and chain rules,

$$\begin{aligned} u_t &= ae^{at}v - ce^{at}v_x + e^{at}v_t \\ &= e^{at}\left(av - cv_x + v_t\right), \\ u_x &= e^{at}v_x, \\ u_{xx} &= e^{at}v_{xx}. \end{aligned}$$

  It follows that

$$\begin{aligned} u_t + cu_x - ku_{xx} - au &= e^{at}\left(av - cv_x + v_t + cv_x - kv_{xx} - av\right) \\ &= e^{at}\left(v_t - kv_{xx}\right) \end{aligned}$$

  which shows that u is a solution of the original PDE if and only if v is a solution of the heat equation $v_t = kv_{xx}$.

- (b) Since $u(x,0) = v(x,0)$, we have $v(x,0) = \phi(x)$. It follows that v is given by the solution of the initial value problem for the heat equation,

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy.$$

Therefore

$$\begin{aligned} u(x,t) &= e^{at}\int_{-\infty}^{\infty} S(x-ct-y)\phi(y)\,dy \\ &= \frac{e^{at}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-(x-y-ct)^2/4kt}\phi(y)\,dy. \end{aligned}$$

5. [20%] Suppose that vehicles moving in one direction down a freeway have a density $u(x,t)$ (measured e.g. in vehicles/mile). Also suppose that at density $0 \le u \le k$ the vehicles move at speed

$$V(u) = \frac{m}{k}(k-u)$$

(measured e.g. in miles/hour) where m and k are constants.

(a) Give an interpretation of the constants m, k.

(b) How many vehicles are there in a section of the road with $a \le x \le b$? What is the rate (measured e.g. in vehicles/hour) at which vehicles cross some point x on the road?

(c) Assume there are no exits or entrances. Write down an equation that expresses "conservation of vehicles" and derive a PDE for $u(x,t)$. Is this PDE linear or nonlinear?

Solution.

- (a) The constant m is the maximum speed of vehicles on an empty road (e.g. m = speed limit + 10 m.p.h.). The constant k is the maximum density of vehicles at which traffic stops moving.

- (b) We have

$$\begin{aligned}\text{number of vehicles in } a \le x \le b &= \int_a^b u(x,t)\,dx,\\ \text{rate at which vehicles cross } x &= \text{density} \times \text{velocity}\\ &= u(x,t)V\left(u(x,t)\right)\end{aligned}$$

- (c) If no vehicles enter or leave the road, then

$$\begin{aligned}&\text{rate of change of number of vehicles in } a \le x \le b\\ &\qquad = \text{rate at which vehicles enter through } a \text{ and leave through } b,\end{aligned}$$

  which implies that

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = u(a,t)V\left(u(a,t)\right) - u(b,t)V\left(u(b,t)\right).$$

Writing

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \int_a^b u_t(x,t)\,dx,$$

$$u(a,t)V\left(u(a,t)\right) - u(b,t)V\left(u(b,t)\right) = -\int_a^b \left[uV(u)\right]_x\,dx,$$

and combining the integrals, we get that

$$\int_a^b \left\{u_t + \left[uV(u)\right]_x\right\}\,dx = 0.$$

Since this equation holds for all $a < b$, the integrand must be zero (assuming that it's a continuous function) and we find that $u(x,t)$ satisfies the PDE

$$u_t + \left[uV(u)\right]_x = 0,$$

or, after use of the explicit expression for V,

$$u_t + \frac{m}{k}\left[u(k-u)\right]_x = 0.$$

This equation can also be written as

$$u_t + mu_x - \frac{2m}{k}uu_x = 0$$

- This equation is nonlinear (because of the uu_x term).