Midterm 2: Sample questions
Math 118A, Fall 2013

1. Find all separated solutions $u(r,t) = F(r)G(t)$ of the radially symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Solve for $G(t)$ explicitly. Write down an ODE for $F(r)$ but don't try to solve it. (What makes the ODE hard to solve explicitly? How many linearly independent solutions for F are there?)

2. Find all separated solutions of the heat equation

$$u_t = ku_{xx}$$

on $0 \leq x \leq L$ that satisfy the mixed Dirichlet-Neumann boundary conditions

$$u(0,t) = 0, \qquad u_x(L,t) = 0.$$

3. Suppose that the function $f(x) = x^2$ is expanded in: (i) a Fourier sine series on $0 < x < 1$; (ii) a Fourier cosine series on $0 < x < 1$; (iii) a full Fourier series on $0 < x < 2$.

(a) Write down the corresponding Fourier series. (b) Give expressions for the corresponding Fourier coefficients as integrals (you don't need to evaluate them). (c) Sketch graphs of the sums of the Fourier series for $-2 < x < 4$.

4. Recall the orthogonality relations for the functions $\sin(n\pi x)$ on $0 \leq x \leq 1$, where $n = 1, 2, 3, \ldots$:

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 1/2 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

If a function $f(x)$, defined on $0 \leq x \leq 1$, has the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

show that

$$\int_0^1 f^2(x)\,dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.$$

5. Solve the following IBVP for $u(x,t)$ in $0 \le x \le L$, $t \ge 0$:

$$\begin{array}{lll} u_{tt} = c^2 u_{xx} & & 0 < x < L, t > 0 \\ u_x(0,t) = 0, & u_x(L,t) = 0 & t \ge 0 \\ u(x,0) = 0, & u_t(x,0) = f(x) & 0 \le x \le L \end{array}$$

Give a physical interpretation of this problem.

6. Solve the following BVP for $u(x,y)$ on the square $0 \le x \le 1$, $0 \le y \le 1$:

$$\begin{array}{lll} u_{xx} + u_{yy} = 0 & & 0 < x < 1, 0 < y < 1 \\ u(0,y) = 0, & u(1,y) = y & 0 \le y < 1 \\ u(x,0) = 0, & & 0 \le x \le 1, \\ u(x,1) = 0, & & 0 \le x \le 1. \end{array}$$

Compute the coefficients in your solution explicitly.