Midterm 2: Sample solutions
Math 118A, Fall 2013

1. Find all separated solutions $u(r,t) = F(r)G(t)$ of the radially symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Solve for $G(t)$ explicitly. Write down an ODE for $F(r)$ but don't try to solve it. (What makes the ODE hard to solve explicitly? How many linearly independent solutions for F are there?)

Solution.

- Using $u(r,t) = F(r)G(t)$ in the equation, we get

  $$F\dot{G} = \frac{k}{r}(rF')'G,$$

  where dots denote t-derivatives and primes denote r-derivatives. Separation of variables gives

  $$\frac{\dot{G}}{kG} = \frac{(rF')'}{rF} = -\lambda$$

  where λ is a separation constant.

- The ODE for $G(t)$ is

  $$\dot{G} = -k\lambda G,$$

  whose solution is

  $$G(t) = Ce^{-k\lambda t}$$

  where C is an arbitrary constant.

- The ODE for $F(r)$ is

  $$(rF')' + \lambda rF = 0,$$

  or

  $$rF'' + F' + \lambda rF = 0.$$

- This is a second order ODE, so it has two linearly independent solutions. Although it's linear, it has variable coefficients (and it isn't an Euler equation), and it can't be solved in terms of elementary functions. When λ is normalized to 1, this ODE is called Bessel's equation of order 0, and its solutions are Bessel functions of order 0.

2. Find all separated solutions of the heat equation

$$u_t = ku_{xx}$$

on $0 \le x \le L$ that satisfy the mixed Dirichlet-Neumann boundary conditions

$$u(0,t) = 0, \qquad u_x(L,t) = 0.$$

Solution.

- Looking for separated solutions $u(x,t) = F(x)G(t)$, we find that

  $$\frac{\dot{G}}{kG} = \frac{F''}{F} = -\lambda$$

  where λ is a separation constant.

- The ODE for $G(t)$ is

  $$\dot{G} = -k\lambda G.$$

  Up to an arbitrary constant factor, the solution is

  $$G(t) = e^{-k\lambda t}.$$

- The eigenvalue problem for F is

  $$F'' + \lambda F = 0, \qquad F(0) = 0, \quad F'(L) = 0.$$

  We consider the following three cases: (i) $\lambda = -\mu^2 < 0$; (ii) $\lambda = 0$; (iii) $\lambda = \mu^2 > 0$. We assume that $\mu > 0$ without loss of generality,

- (i) If $\lambda = -\mu^2$, then $F'' - \mu^2 F = 0$ and

  $$F(x) = A\cosh\mu x + B\sinh\mu x.$$

  The BC $F(0) = 0$ implies that $A = 0$, and the BC $F'(L) = 0$ then implies that $\mu B\cosh\mu L = 0$. Since $\mu\cosh\mu L \neq 0$, we get that $B = 0$, so $F = 0$ and λ is not an eigenvalue.

- (ii) If $\lambda = 0$, then $F'' = 0$, so $F(x) = A + Bx$. The BC $F(0) = 0$ implies that $A = 0$, and the BC $F'(L) = 0$ implies that $B = 0$, so $F = 0$ and λ is not an eigenvalue.

- (iii) If $\lambda = \mu^2 > 0$, then $F'' + \mu^2 F = 0$ and

  $$F(x) = A \cos \mu x + B \sin \mu x.$$

  The BC $F(0) = 0$ implies that $A = 0$, and then the BC $F'(L) = 0$ implies that $\mu B \cos \mu L = 0$. We have a solution with $B \neq 0$ if $\cos \mu L = 0$ or $\mu = \mu_n$ where

  $$\mu_n = \left(n - \frac{1}{2}\right) \frac{\pi}{L} \qquad \text{for } n = 1, 2, \ldots.$$

  In that case, $\lambda = \mu_n^2$ is an eigenvalue with eigenfunction $\sin \mu_n x$.

- Up to an arbitrary constant factor, the separated solutions are therefore

  $$u(x,t) = \sin \left[\left(n - \frac{1}{2}\right) \frac{\pi x}{L}\right] \exp \left[-k \left(n - \frac{1}{2}\right)^2 \frac{\pi^2 t}{L^2}\right]$$

  where $n = 1, 2, 3, \ldots$.

3. Suppose that the function $f(x) = x^2$ is expanded in: (i) a Fourier sine series on $0 < x < 1$; (ii) a Fourier cosine series on $0 < x < 1$; (iii) a full Fourier series on $0 < x < 2$.

(a) Write down the corresponding Fourier series. (b) Give expressions for the corresponding Fourier coefficients as integrals (you don't need to evaluate them). (c) Sketch graphs of the sums of the Fourier series for $-2 < x < 4$.

Solution.

- (a)–(b) The Fourier sine series in $0 < x < 1$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x), \qquad b_n = 2\int_0^1 x^2 \sin(n\pi x)\, dx.$$

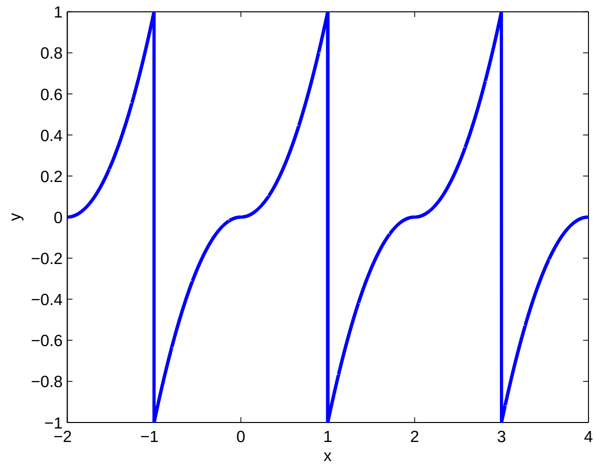
  The Fourier cosine series in $0 < x < 1$ is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x), \qquad a_n = 2\int_0^1 x^2 \cos(n\pi x)\, dx.$$

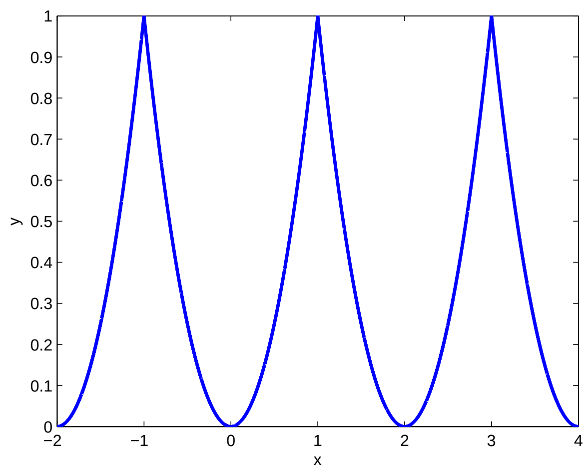
  The full Fourier series in $0 < x < 2$ is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\}$$
$$a_n = \int_0^2 x^2 \cos(n\pi x)\, dx, \quad b_n = \int_0^2 x^2 \sin(n\pi x)\, dx.$$

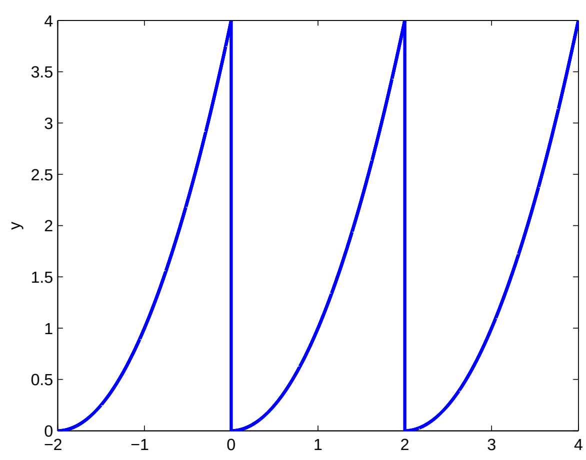
- For $-\infty < x < \infty$, the Fourier sine and cosine series converge to the odd and even periodic extensions of x^2 on $0 < x < 1$, respectively. The full Fourier series converges to the periodic extension of x^2 on $0 < x < 2$. The graphs are shown on the next page.



(a) Sum of the sine series.



(b) Sum of the cosine series.



(c) Sum of the full Fourier series.

4. Recall the orthogonality relations for the functions $\sin(n\pi x)$ on $0 \le x \le 1$, where $n = 1, 2, 3, \ldots$:

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 1/2 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

If a function $f(x)$, defined on $0 \le x \le 1$, has the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

show that

$$\int_0^1 f^2(x)\,dx = \frac{1}{2}\sum_{n=1}^{\infty} b_n^2.$$

Solution.

- We have

$$\begin{aligned}
\int_0^1 f^2(x)\,dx &= \int_0^1 \left[\sum_{m=1}^{\infty} b_m \sin(m\pi x)\right]\left[\sum_{n=1}^{\infty} b_n \sin(n\pi x)\right] dx \\
&= \int_0^1 \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \sin(m\pi x)\sin(n\pi x)\right] dx \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_m b_n \left[\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx\right] \\
&= \frac{1}{2}\sum_{n=1}^{\infty} b_n^2,
\end{aligned}$$

  since the only nonzero terms in the series are the ones with $m = n$, in which case the integrals are $1/2$.

- This result is called Parseval's theorem. It can be interpreted as saying that the "energy" of f can be computed spatially by integrating f^2 or spectrally by summing the squares b_n^2 of its Fourier components.

5. Solve the following IBVP for $u(x,t)$ in $0 \le x \le L$, $t \ge 0$:

$$\begin{array}{lll} u_{tt} = c^2 u_{xx} & & 0 < x < L, t > 0 \\ u_x(0,t) = 0, & u_x(L,t) = 0 & t \ge 0 \\ u(x,0) = 0, & u_t(x,0) = f(x) & 0 \le x \le L \end{array}$$

Give a physical interpretation of this problem.

Solution.

- Looking for separated solutions $u(x,t) = F(x)G(t)$, we get

  $$F\ddot{G} = c^2 F'' G$$

  which implies that

  $$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = -\lambda$$

  where λ is a separation constant.

- The eigenvalue problem for $F(x)$ in $0 < x < L$ is

  $$F'' + \lambda F = 0, \qquad F'(0) = 0, \quad F'(L) = 0.$$

  The eigenfunctions and eigenvalues are

  $$F(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad \lambda = \left(\frac{n\pi}{L}\right)^2$$

  for $n = 0, 1, 2, 3, \ldots$.

- The corresponding ODE for $G(t)$ is

  $$\ddot{G} + \left(\frac{n\pi c}{L}\right)^2 G = 0.$$

  If $n \ge 1$, the general solution is

  $$G(t) = a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)$$

  where a_n, b_n are arbitrary constants. If $n = 0$, then the general solution is $G(t) = a_0 + b_0 t/2$.

- Superposing these separated solutions, we get the general solution of the PDE and the BCs

$$u(x,t) = a_0 + \frac{1}{2}b_0 t + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

- Imposing the initial condition for $u(x,0)$, we get

$$0 = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

which implies that $a_n = 0$ for every n. Thus

$$u(x,t) = \frac{1}{2}b_0 t + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi ct}{L}\right)\cos\left(\frac{n\pi x}{L}\right) \tag{1}$$

- It follows that

$$u_t(x,t) = \frac{1}{2}b_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \cos\left(\frac{n\pi ct}{L}\right)\cos\left(\frac{n\pi x}{L}\right).$$

Imposing the initial condition for $u_t(x,0)$, we get

$$f(x) = \frac{1}{2}b_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \cos\left(\frac{n\pi x}{L}\right),$$

and the formula for Fourier cosine coefficients gives

$$b_0 = \frac{2}{L}\int_0^L f(x)\,dx, \qquad \frac{n\pi c}{L} b_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

- The solution is therefore given by (1) with coefficients

$$b_0 = \frac{2}{L}\int_0^L f(x)\,dx, \quad b_n = \frac{2}{n\pi c}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \quad n = 1, 2, \dots.$$

- This problem describes the vibrations of an elastic string with initial displacement zero and initial velocity f. The ends of the string are free to slide up or down. Note that the average of the solution grows linearly in time with a velocity $b_0/2$ that's equal to the average of the initial velocity.

6. Solve the following BVP for $u(x, y)$ on the square $0 \leq x \leq 1$, $0 \leq y \leq 1$:

$$\begin{aligned} &u_{xx} + u_{yy} = 0 && 0 < x < 1, 0 < y < 1 \\ &u(0, y) = 0, \quad u(1, y) = y && 0 \leq y < 1 \\ &u(x, 0) = 0, \quad u(x, 1) = 0, && 0 \leq x \leq 1. \end{aligned}$$

Compute the coefficients in your solution explicitly.

Solution.

- We look for separated solutions

$$u(x, y) = F(x)G(y).$$

  Then $F''G + FG'' = 0$, where primes denote derivatives with respect to x or y as appropriate, which implies that

$$\frac{F''}{F} = -\frac{G''}{G} = \lambda$$

  where λ is a separation constant.

- We impose BCs in y, since both are homogeneous, which gives

$$G'' + \lambda G = 0, \qquad G(0) = 0, \quad G(1) = 0.$$

  The eigenfunctions and eigenvalues are

$$G(y) = \sin(n\pi y), \qquad \lambda = n^2\pi^2 \qquad \text{for } n = 1, 2, 3, \ldots.$$

- The corresponding ODE for F, with homogeneous BC at $x = 0$, is

$$F'' - n^2\pi^2 F = 0, \qquad F(0) = 0.$$

  It follows that, up to an arbitrary constant factor,

$$F(x) = \sinh(n\pi x),$$

  and the separated solutions are therefore $u(x, y) = \sinh(n\pi x)\sin(n\pi y)$.

- The general solution satisfying the homogeneous PDE and BCs is

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi x)\sin(n\pi y).$$

Imposing the nonhomogeneous BC at $x = 1$, we get

$$y = \sum_{n=1}^{\infty} b_n \sinh(n\pi)\sin(n\pi y) \qquad \text{for } 0 < y < 1.$$

- Using the formula for Fourier sine coefficients and integrating by parts, we get

$$\begin{aligned} b_n \sinh(n\pi) &= 2\int_0^1 y\sin(n\pi y)\,dy \\ &= \frac{2}{n\pi}\left[-y\cos(n\pi y)\right]_0^1 + \frac{2}{n\pi}\int_0^1 1\cdot\cos(n\pi y)\,dy \\ &= -\frac{2}{n\pi}\cos(n\pi) + \frac{2}{(n\pi)^2}\left[\sin(n\pi y)\right]_0^1 \\ &= \frac{2(-1)^{n+1}}{n\pi}. \end{aligned}$$

- In summary, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi\sinh(n\pi)}\sinh(n\pi x)\sin(n\pi y).$$