

**Midterm 2: Solutions**  
**Math 118A, Fall 2013**

1. [25%] Find all separated solutions  $u(x, t) = F(x)G(t)$  of the advection equation

$$u_t + cu_x = 0$$

where  $c$  is a constant. Show that the separated solutions have the same form as the general solution  $u(x, t) = f(x - ct)$  for a suitable function  $f$ .

**Solution.**

- Using the separated solution in the PDE, we get

$$F\dot{G} + cF'G = 0.$$

Separation of variables gives

$$\frac{F'}{F} = -\frac{\dot{G}}{cG} = \lambda$$

where  $\lambda$  is a separation constant.

- The ODE for  $F(x)$  is  $F' = \lambda F$ , whose solution is  $F(x) = Ce^{\lambda x}$ , where  $C$  is a constant of integration.
- The ODE for  $G$  is  $\dot{G} + \lambda cG = 0$ , whose solution is  $G(t) = Ce^{-\lambda ct}$ .
- Thus, up to an arbitrary constant factor, the separated solutions are

$$u(x, t) = e^{\lambda x} e^{-c\lambda t}.$$

- This solution can be written as  $u(x, t) = e^{\lambda(x-ct)}$ , which agrees with the general solution with  $f(\xi) = e^{\lambda\xi}$ .

2. [25%] Let  $f$  be the constant function  $f(x) = 1$  defined on the interval  $0 \leq x \leq 1$ .

(a) Compute the Fourier sine series of  $f(x)$  on  $0 \leq x \leq 1$ . (Evaluate the Fourier sine coefficients explicitly.)

(b) Compute the Fourier cosine series of  $f(x)$  on  $0 \leq x \leq 1$ . (Evaluate the Fourier cosine coefficients explicitly.)

(c) On the next page, sketch the sums of the Fourier sine and cosine series of  $f(x)$  for  $-2 \leq x \leq 4$ .

**Solution.**

- (a) The Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) \quad 0 < x < 1$$

where

$$\begin{aligned} b_n &= 2 \int_0^1 1 \cdot \sin(n\pi x) dx \\ &= -\frac{2}{n\pi} [\cos(n\pi x)]_0^1 \\ &= \frac{2}{n\pi} [1 - (-1)^n] \\ &= \begin{cases} 4/n\pi & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

- The Fourier sine series of  $f$  is therefore

$$\begin{aligned} f(x) &= \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin[(2n-1)\pi x]. \end{aligned}$$

- (b) The Fourier cosine series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) \quad 0 < x < 1$$

where

$$a_n = 2 \int_0^1 1 \cdot \cos(n\pi x) dx.$$

Therefore,

$$a_0 = 2 \int_0^1 1 dx = 2,$$

and

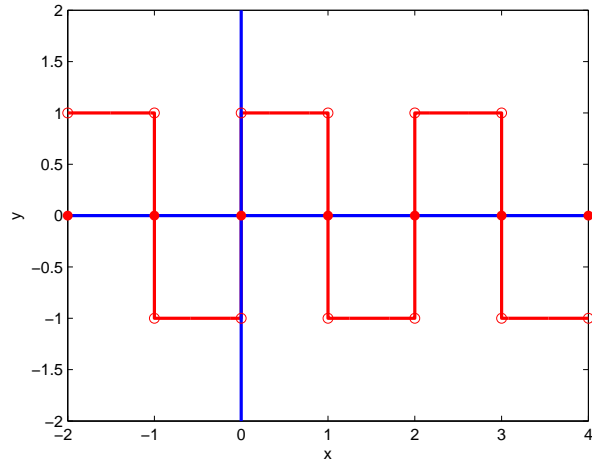
$$a_n = 2 \int_0^1 \cos(n\pi x) dx = \frac{2}{n\pi} [\sin(n\pi x)]_0^1 = 0 \quad n \geq 1.$$

(i.e., 1 and  $\cos n\pi x$  are orthogonal.)

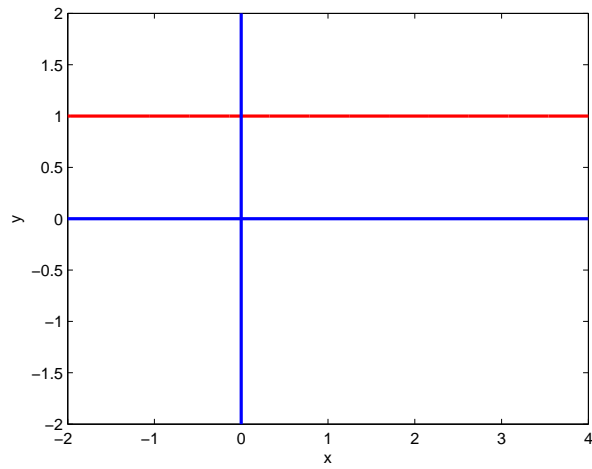
- The Fourier cosine series of  $f$  is therefore just

$$f(x) = 1.$$

- (c) The Fourier sine series converges to the odd extension of  $f$  (see the graph). At points where this function has jump discontinuities, the Fourier series converges to the average values of the left and right limits, which in this case is 0.
- The Fourier cosine series converges to the even periodic extension of  $f$ , which is just 1.



(a) Sum of the Fourier sine series of 1 on  $0 < x < 1$ .



(b) Sum of the Fourier cosine series of 1 on  $0 < x < 1$ .

3. [20%] Suppose that  $f(x)$  is a twice continuously differentiable function defined on the interval  $0 \leq x \leq \pi$  such that  $f(0) = 0$ ,  $f(\pi) = 0$ . Let  $a_n$  be the  $n$ th Fourier sine coefficient of  $f$  and  $b_n$  the  $n$ th Fourier sine coefficient of the second derivative  $f''$  of  $f$ ,

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx, \quad b_n = \frac{2}{\pi} \int_0^\pi f''(x) \sin(nx) dx.$$

Express  $b_n$  in terms of  $a_n$ . (Hint: Integration by parts.)

**Solution.**

- Integrating by parts twice in the expression for  $b_n$  to take derivatives off  $f$  and put them on  $\sin nx$  or  $\cos nx$ , we get

$$\begin{aligned} b_n &= \frac{2}{\pi} [f'(x) \sin(nx)]_0^\pi - \frac{2}{\pi} \int_0^\pi f'(x) \cdot n \cos(nx) dx \\ &= 0 - \frac{2n}{\pi} \left\{ [f(x) \cdot \cos(nx)]_0^\pi - \int_0^\pi f(x) \cdot (-n) \sin(nx) dx \right\} \\ &= -\frac{2n}{\pi} [f(\pi) \cos n - f(0)] - n^2 \cdot \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx \end{aligned}$$

- Since we assume that  $f(0) = f(\pi) = 0$ , we get that

$$b_n = -n^2 a_n$$

**Remark.** This question show that taking the second derivative of  $f$  corresponds to multiplying its Fourier coefficients by  $-n^2$ . Thus, Fourier series (or Fourier transforms) convert differentiation into an algebraic operation (multiplication by  $n$ ). In particular, Fourier analysis enables us to solve constant-coefficient, linear differential equations by converting them into algebraic equations.

4. [30%] (a) Use separation of variables to solve the following initial-boundary value problem for  $u(x, t)$  in  $0 < x < L, t > 0$ :

$$\begin{aligned} u_t &= ku_{xx} + cu & 0 < x < L, t > 0 \\ u(0, t) &= 0, \quad u(L, t) = 0, & t > 0 \\ u(x, 0) &= f(x), & 0 \leq x \leq L \end{aligned}$$

where  $k, c$  are positive constants.

- (b) Give a physical interpretation of this problem.  
(c) Discuss the behavior of the solution as  $t \rightarrow +\infty$ .

**Solution.**

- (a) We look for separated solutions of the form  $u(x, t) = F(x)G(t)$ . Then

$$F\dot{G} = kF''G + cFG.$$

Dividing by  $FG$  and separating variables, we get

$$\frac{F''}{F} = \frac{\dot{G}}{kG} - \frac{c}{k} = -\lambda,$$

where  $\lambda$  is a separation constant. (The separation constant could be defined in other ways, but the final result would be the same. The choice here is the simplest one for writing the eigenvalue problem.)

- The eigenvalue problem for  $F(x)$  is

$$F'' + \lambda F, \quad F(0) = 0, \quad F(L) = 0.$$

The eigenvalues and eigenfunctions are

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \dots \quad (1)$$

- The ODE for  $G$  is

$$\dot{G} + (k\lambda - c)G = 0.$$

Up to an arbitrary constant factor, the solution is

$$G(t) = e^{-(k\lambda - c)t}.$$

The separated solutions are therefore

$$u(x, t) = \sin\left(\frac{n\pi x}{L}\right) e^{-(k\lambda_n - c)t}.$$

- Taking a linear combination of the separated solutions, we get that the general solution of the PDE and BCs is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-(k\lambda_n - c)t}. \quad (2)$$

Imposition of the initial condition at  $t = 0$  gives

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

so  $b_n$  is the  $n$ th Fourier coefficient of  $f(x)$ ,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx. \quad (3)$$

- In summary, the solution of the IBVP is given by (2), where  $\lambda_n$  is given in (1), and  $b_n$  is given in (3).
- (b) This problem describes the flow of heat in a rod with a heat source whose strength is proportional to the temperature  $u$ . Both endpoints of the rod are held at a fixed temperature 0, and the initial temperature is  $f(x)$ .
- (c) The  $n$ th Fourier mode decays in time if  $c < k\lambda_n$  and grows in time if  $c > k\lambda_n$ . The slowest decaying or fastest growing mode is the first mode with  $n = 1$ . Suppose for definiteness that  $b_1 > 0$ . Then the solution decays to 0 as  $t \rightarrow \infty$  if  $c < k\lambda_1$ , and grows to  $\infty$  as  $t \rightarrow \infty$  if  $c > k\lambda_1$ . If  $c = k\lambda_1$ , then  $u(x, t) \rightarrow b_1 \sin(\pi x)$  approaches a steady state as  $t \rightarrow \infty$ .
- Let's introduce a dimensionless parameter

$$R = \frac{\pi^2 c}{k\lambda_1} = \frac{cL^2}{k}.$$

If  $R < \pi^2$ , the temperature decays because heat leaks out the ends of the rod at a faster rate than it is generated by the source. This happens if the source-coefficient  $c$  or the length  $L$  of the rod are sufficiently small, or if the diffusivity  $k$  is sufficiently large (which makes sense physically). On the other hand if  $R > \pi^2$ , the source generates heat at a faster rate than it can leak out the ends and the temperature grows.