Partial Differential Equations
Math 118B, Winter 2014
Final: Solutions

1. [30 pts] (a) State Green's first identity.
(b) Suppose that the nonzero function $u(x)$ is a eigenfunction of the Laplacian with eigenvalue $-\infty < \lambda < \infty$ in a bounded region Ω with smooth boundary $\partial\Omega$ and Dirichlet boundary conditions, meaning that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Show that $\lambda > 0$.

Solution.

- (a) If $u, v : \bar{\Omega} \to \mathbb{R}$ are continuously differentiable functions on a bounded open set Ω with smooth boundary $\partial\Omega$, then

$$\int_\Omega u\Delta v\, dx = -\int_\Omega \nabla u \cdot \nabla v\, dx + \int_{\partial\Omega} u\frac{\partial v}{\partial n}\, dS.$$

  (This identity follows from an application of the divergence theorem to the vector identity $\nabla \cdot (u\nabla v) = u\Delta v + \nabla u \cdot \nabla v$.)

- (b) If $-\Delta u = \lambda u$ and $u = 0$ on $\partial\Omega$, then Green's first identity with $u = v$ gives

$$-\lambda \int_\Omega u^2\, dx = -\int_\Omega |\nabla u|^2\, dx.$$

  Since $u \neq 0$ and $u^2 \geq 0$, it follows that $\int_\Omega u^2\, dx > 0$, and therefore

$$\lambda = \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega u^2\, dx} \geq 0.$$

- If $\lambda = 0$, then $\int_\Omega |\nabla u|^2\, dx = 0$, so $\nabla u = 0$ in Ω, and therefore $u =$ constant. Since $u = 0$ on $\partial\Omega$, it follows that $u = 0$ in Ω, so $\lambda = 0$ is not an eigenvalue, and $\lambda > 0$.

Remark. We didn't have time to discuss eigenvalue problems for the Laplacian. One way these problems arise is from looking for separated solutions of the wave equation

$$\phi_{tt} = c^2\Delta\phi$$

of the form

$$\phi(x,t) = u(x)e^{-i\omega t}.$$

For example, a two-dimensional version of the corresponding Dirichlet problem describes the time-periodic vibrations of a drum of shape Ω fixed at its boundary $\partial\Omega$. In that case, $\lambda = \omega^2/c^2 > 0$ where $\omega > 0$ is a resonant frequency of the drum.

Mark Kac (1966) wrote a famous paper entitled "Can you hear the shape of a drum?" which asked if you can determine the shape of a region $\Omega \subset \mathbb{R}^2$ from its eigenvalues λ. The answer is you can't: there are regions with different shapes whose Dirichlet Laplacians have exactly the same eigenvalues.

2. [30 pts] Consider the Dirichlet problem for $u(x,y,z)$ for the three-dimensional Laplacian Δ in the upper-half space $z>0$:

$$\begin{aligned} &\Delta u = 0 && \text{in } z>0, \\ &u(x,y,0) = h(x,y) && \text{on } z=0, \\ &u(x,y,z)\to 0 && \text{as } z\to\infty, \end{aligned} \tag{1}$$

where $h(x,y)$ is a continuous function that is zero when $|(x,y)|$ is sufficiently large.

(a) If $\vec{x}=(x,y,z)$ and $\vec{\xi}=(\xi,\eta,\zeta)$ with $\zeta>0$, verify that the Green's function for this problem corresponding to a point source at $\vec{\xi}$ is

$$G(\vec{x};\vec{\xi}) = \frac{1}{4\pi|\vec{x}-\vec{\xi}|} - \frac{1}{4\pi|\vec{x}-\vec{\xi}^*|}$$

where $\vec{\xi}^* = (\xi,\eta,-\zeta)$.

Hint. You can assume, without proof, that $G_F(\vec{x}) = 1/4\pi|\vec{x}|$ is the free space Green's function for the Laplacian, such that $-\Delta G_F = \delta(\vec{x})$.

(b) State Green's second identity. Use this identity and a formal calculation with δ-functions to derive an integral representation of the solution $u(\xi,\eta,\zeta)$ of (1). Write your answer as an integral with respect to x, y.

Solution.

- (a) We have

$$\begin{aligned} -\Delta G &= -\Delta\left(\frac{1}{4\pi|\vec{x}-\vec{\xi}|}\right) + \Delta\left(\frac{1}{4\pi|\vec{x}-\vec{\xi}^*|}\right) \\ &= \delta(\vec{x}-\vec{\xi}) - \delta(\vec{x}-\vec{\xi}^*) \\ &= \delta(\vec{x}-\vec{\xi}) \end{aligned}$$

  in $z>0$, since $\vec{x}\neq\vec{\xi}^*$ in the upper-half space.

- We have $G=0$ on $z=0$, since in that case

$$|\vec{x}-\vec{\xi}| = \sqrt{(x-\xi)^2+(y-\eta)^2+\zeta^2} = |\vec{x}-\vec{\xi}^*|.$$

  Also, $G(\vec{x};\vec{\xi})\to 0$ as $z\to\infty$, so G is the Green's function for this problem.

- (b) Green's second identity states that

$$\int_{\Omega} (u\Delta v - v\Delta u)\, d\vec{x} = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS.$$

  (This identity follows from an application of the divergence theorem to the vector identity $\nabla \cdot (u\nabla v - v\nabla u) = u\Delta v - v\Delta u$.)

- Applying this identity with $v(\vec{x}) = G(\vec{x}; \vec{\xi})$ and Ω equal to the upper-half space, and using the equations satisfied by u and G, we get

$$\int_{\Omega} -u(\vec{x})\delta(\vec{x} - \xi)\, d\vec{x} = \int_{\partial\Omega} h(\vec{x})\frac{\partial G(\vec{x}; \vec{\xi})}{\partial n(\vec{x})}\, dS(\vec{x}).$$

  Here, we assume that $u(\vec{x})$ decays sufficiently rapidly as $|\vec{x}| \to \infty$ for us to neglect any contribution from the boundary terms at infinity. Using the substitution property of the delta-function, we get

$$u(\vec{\xi}) = -\int_{\partial\Omega} h(\vec{x})\frac{\partial G(\vec{x}; \vec{\xi})}{\partial n(\vec{x})}\, dS(\vec{x}).$$

- The outward normal derivative on $z = 0$ is $\partial/\partial n = -\partial/\partial z$ and $dS = dxdy$. We also compute that

$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = \frac{-\zeta}{2\pi \left[(x-\xi)^2 + (y-\eta)^2 + \zeta^2\right]^{3/2}}.$$

  It follows that

$$u(\xi, \eta, \zeta) = \frac{\zeta}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{h(x, y)}{\left[(x-\xi)^2 + (y-\eta)^2 + \zeta^2\right]^{3/2}}\, dxdy.$$

3. [30 pts] (a) Define the derivative f' of a distribution f on $\mathbb{R}$.
(b) For $\epsilon > 0$, let

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } |x| > \epsilon, \\ -1/\epsilon^2 & \text{if } -\epsilon < x < 0 \\ 1/\epsilon^2 & \text{if } 0 < x < \epsilon. \end{cases}$$

Use your definition in (a) to compute the distributional derivative f'_ϵ.
(c) Show that $f_\epsilon \to f$ weakly as $\epsilon \to 0$ to some distribution f and find f.

Solution.

- (a) If $f : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ is a distribution (a continuous linear functional on the space $\mathcal{D}(\mathbb{R})$ of test functions), then $f' : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ is defined by

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle \qquad \text{for all } \phi \in \mathcal{D}(\mathbb{R}).$$

- (b) Using the definition of the distributional derivative, the fundamental theorem of calculus, and the definition of the δ-function, we have

$$\begin{aligned} \langle f'_\epsilon, \phi \rangle &= -\langle f_\epsilon, \phi' \rangle \\ &= -\frac{1}{\epsilon^2} \int_{-\epsilon}^{0} \phi'(x)\, dx + \frac{1}{\epsilon^2} \int_0^\epsilon \phi'(x)\, dx \\ &= \frac{1}{\epsilon^2} \left[\phi(\epsilon) - 2\phi(0) + \phi(-\epsilon) \right] \\ &= \langle \frac{1}{\epsilon^2} \left[\delta_\epsilon - 2\delta + \delta_{-\epsilon} \right], \phi \rangle. \end{aligned}$$

  It follows that

$$f'_\epsilon = \frac{1}{\epsilon^2} \left[\delta_\epsilon - 2\delta + \delta_{-\epsilon} \right]$$

- Here, we use the notation δ_a to denote the δ-function supported at a, which is defined by

$$\langle \delta_a, \phi \rangle = \phi(a).$$

  In particular, $\delta_0 = \delta$. This distribution can also be written (a bit more formally) as $\delta_a(x) = \delta(x - a)$.

- (c) *Note.* There was a typo in the question, which asked for the limit of f_ϵ as $\epsilon \to \infty$. (That's almost always going to be a typo in mathematics.) It's not too hard to show that $f_\epsilon \to 0$ as $\epsilon \to \infty$. We give the solution for the more difficult, but more interesting, intended question, which is to find the limit as $\epsilon \to 0^+$, leading to a dipole.

- For $\phi \in \mathcal{D}(\mathbb{R})$, we have as $\epsilon \to 0^+$ that

$$\begin{aligned} \langle f_\epsilon, \phi \rangle &= \frac{1}{\epsilon^2} \left[\int_0^\epsilon \phi(x)\, dx - \int_{-\epsilon}^0 \phi(x)\, dx \right] \\ &= \frac{1}{\epsilon^2} \int_0^\epsilon \left[\phi(x) - \phi(-x) \right]\, dx \\ &= \frac{1}{\epsilon^2} \int_0^\epsilon \phi'(\xi) \cdot 2x\, dx \\ &\to \frac{\phi'(0)}{\epsilon^2} \int_0^\epsilon 2x\, dx \\ &= \phi'(0) = \langle -\delta', \phi \rangle. \end{aligned}$$

  Here, we change $x \mapsto -x$ in the second integral and apply the mean value theorem with $|\xi(x)| < x$, then use the continuity of ϕ' and the definition $\langle \delta', \phi \rangle = -\phi'(0)$ of δ'. It follows that $f_\epsilon \to -\delta'$ weakly as $\epsilon \to 0^+$.

Remark. One can show similarly (in fact more easily) that

$$\frac{1}{\epsilon} \left[\delta\left(x - \frac{\epsilon}{2}\right) - \delta\left(x + \frac{\epsilon}{2}\right) \right] \to -\delta'(x)$$

weakly as $\epsilon \to 0^+$. The distribution $-\delta'$ has the physical interpretation of a dipole, which is obtained by taking the limit as $\epsilon \to 0$ of two point sources of large but opposite strengths $1/\epsilon$, $-1/\epsilon$ that are a small distance ϵ apart (i.e., in the limit when strength $\times$ distance is constant). Although the point source strengths cancel, a finite, nonzero dipole source is left.

The function f_ϵ is an approximation of the source density of a dipole. The derivative f_ϵ' approximates a quadrupole (a similar limit of two dipoles of opposite strengths), whose source density is the distribution $-\delta''(x)$.

Things are a bit more complicated in several space dimensions because dipoles depend on the direction in which the point sources approach each other, and quadrupoles depend on the direction of the dipoles and the direction in which the dipoles approach each other.

4. [30 pts] Suppose that $f(x)$ is a smooth, rapidly decaying function. Use the Fourier transform to solve the following initial value problem for the linearized KdV-Burgers equation for $u(x,t)$:

$$\begin{aligned} &u_t - u_{xx} + u_{xxx} = 0 \qquad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x). \end{aligned}$$

Write your answer as a convolution with a Green's function. (Express the Green's function as a Fourier integral but don't try to invert it.) How does the solution behave as $t \to \infty$?

Solution.

- Let
$$U(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\, dx$$
denote the Fourier transform of $u(x,t)$ with respect to x. Then, since x-derivatives transform to multiplication by ik, we get
$$\begin{aligned} &U_t + k^2 U - ik^3 U = 0 \qquad -\infty < k < \infty, \quad t > 0, \\ &U(k,0) = F(k), \end{aligned}$$
where $F(k)$ is the Fourier transform of $f(x)$.
- The solution of this ODE for U is
$$U(k,t) = F(k)e^{(ik^3-k^2)t}.$$
- By the convolution theorem, $u = f * G$ where G is the inverse Fourier transform of $e^{(ik^3-k^2)t}$ with respect to k, and the convolutions is taken with respect to x. That is,
$$u(x,t) = \int_{-\infty}^{\infty} f(x-y)G(y,t)\, dy$$
where
$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+(ik^3-k^2)t}\, dk.$$
- Since $|e^{ikx+(ik^3-k^2)t}| = e^{-k^2 t} \to 0$ as $t \to \infty$ for every $k \neq 0$, the solution decays to zero as $t \to \infty$.

Remark. A more detailed argument is actually required to show that we can neglect the slowly-decaying contributions from wavenumbers k close to 0 for large t and conclude that $u(x,t) \to 0$ as $t \to \infty$. We won't give the details, but one can prove that the maximum value of the solution decays like $t^{-1/2}$,

$$|u(x,t)| \leq \frac{C\|f\|}{\sqrt{t}}, \qquad \|f\| = \int_{-\infty}^{\infty} |f(x)|\, dx.$$

This is the same estimate that one gets for the heat equation $u_t - \mu u_{xx} = 0$.

This PDE is the linearization at $u = 0$ of the KdV-Burgers equation

$$u_t + uu_x - \mu u_{xx} + \nu u_{xxx} = 0,$$

which describes the combined effects of nonlinearity (uu_x), dissipation (μu_{xx}), and dispersion (νu_{xxx}).

5. [30 pts] Consider the following IBVP for the Fisher (or KPP) equation on the interval $0 < x < 2\pi$ with homogeneous Dirichlet BCs, where $\lambda > 0$ is a constant parameter:

$$
\begin{aligned}
&u_t = u_{xx} + \lambda u(1-u) \qquad 0 < x < 2\pi, \quad t > 0\\
&u(0,t) = 0, \quad u(2\pi,t) = 0,\\
&u(x,0) = f(x).
\end{aligned}
$$

(a) Determine the linearized stability of the equilibrium solution $u = 0$.

(b) Determine the linearized stability of the equilibrium solution $u = 1$.

(c) Write down an ODE for traveling wave solutions of the Fisher equation. (You don't have to solve the ODE or sketch a phase plane.)

Solution.

- (a) Linearizing the PDE and BC at $u = 0$, we get

  $$u_t = u_{xx} + \lambda u, \qquad u(0,t) = u(2\pi,t) = 0.$$

  (Note that $u(1-u) \sim u$ when u is small.) The separated solutions are

  $$u(x,t) = e^{(\lambda - n^2)t} \sin nx, \qquad n = 1, 2, 3, \dots.$$

  The general solution is an arbitrary linear combination of these separated solution.

- All of the separated solutions decay as $t \to \infty$ as $t \to \infty$ if $\lambda < 1$, and the $n = 1$ solution grows if $\lambda > 1$, so $u = 0$ is linearly stable if $\lambda < 1$ and linearly unstable if $\lambda > 1$.

- (b) *Note.* This part doesn't make sense as stated since $u = 1$ doesn't satisfy the boundary conditions $u = 0$ at $x = 0, 2\pi$, so it's not an equilibrium solution.

- For definiteness, suppose that we consider the PDE with boundary conditions $u = 1$ at $x = 0, 2\pi$. Then, writing $u(x,t) = 1 + u_1(x,t)$ and linearizing the PDE and BC at $u_1 = 0$, we get

  $$u_{1t} = u_{1xx} - \lambda u_1, \qquad u_1(0,t) = u_1(2\pi,t) = 0.$$

  (Note that $u(1-u) = -(1+u_1)u_1 \sim -u_1$ when u_1 is small.) The separated solutions are

  $$u_1(x,t) = e^{-(\lambda + n^2)t} \sin nx, \qquad n = 1, 2, 3, \dots.$$

- All of these separated solutions decay as $t \to \infty$ if $\lambda > 0$, so $u = 1$ is linearly stable for every $\lambda > 0$.
- (c) The traveling wave ODE for $u(x, t) = f(x - ct)$ is

  $$f'' + cf' + \lambda f(1 - f) = 0.$$

  See Figure 1 for the phase plane and a traveling wave solution.

Remark. The KPP is a reaction-diffusion equation. Spatially independent solutions satisfy the logistic ODE $u_t = \lambda u(1-u)$, which describes the growth of a population with limited resources. The equilibrium $u = 0$ is unstable and $u = 1$ is stable; it corresponds to the carrying capacity of the system i.e., the maximum population that can be sustained indefinitely by the available resources.

Traveling wave solutions of the KPP equation on $-\infty < x < \infty$ describe the invasion of the unstable state $u = 0$ by the stable state $u = 1$. Normalizing $\lambda = 1$ without loss of generality for $-\infty < x < \infty$, one can show that non-negative traveling waves ($f \geq 0$) such that $f(\xi) \to 1$ as $\xi \to -\infty$ and $f(\xi) \to 0$ as $\xi \to \infty$ exist for every wave speed $2 \leq c < \infty$.

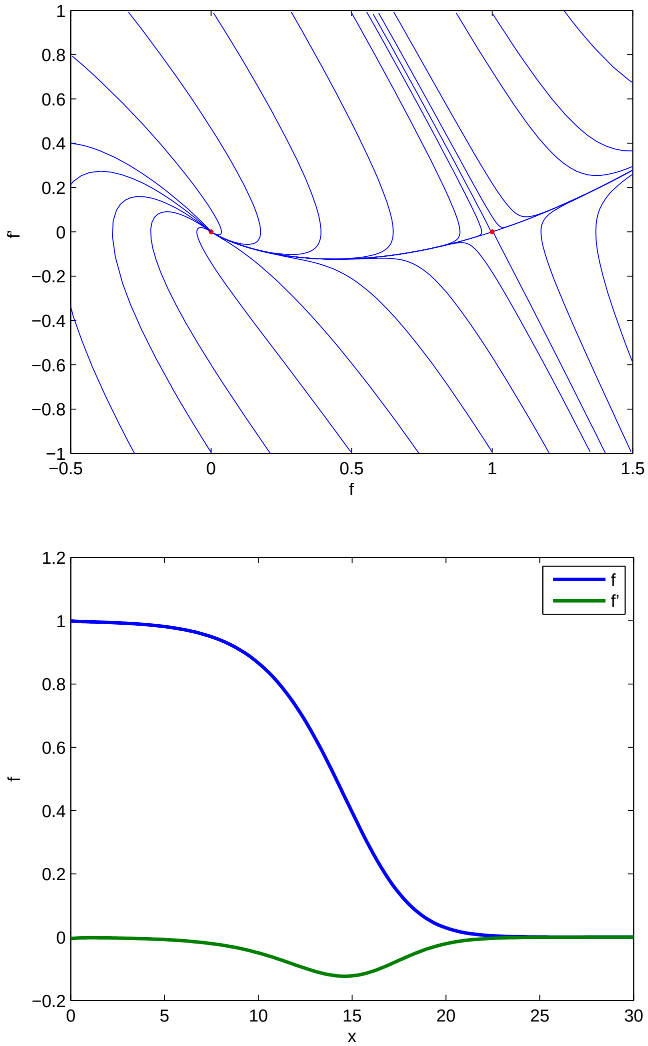

Figure 1: Top. The (f, f') phase-plane for the KPP traveling wave equation $f'' + 2f' + f(1 - f) = 0$. There is a heteroclinic orbit from the saddle point $(1, 0)$ to the stable node $(0, 0)$, which are indicated by the red dots. Bottom. Graph of the corresponding traveling wave profile $f(x)$ and its derivative $f'(x)$ versus x.

6. [30 pts] Consider the following initial value problem for the inviscid Burgers equation with initial data that contains a shock at $x = 1$:

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0,$$

$$u(x,0) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x > 1, \\ x & \text{if } 0 < x < 1. \end{cases}$$

(a) Sketch the characteristics and the approximate location of the shock in the (x,t)-plane.

(b) Solve this initial value problem. How does the strength of the shock behave as $t \to \infty$?

Solution.

- (a) See Figure 2.
- (b) Using the method of characteristics,

  $$\frac{du}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = u,$$

  with $x = \xi$ and $u = u(\xi, 0)$ at $t = 0$, we get that $u = 0$ on $x = \xi$ for $\xi \leq 0$ or $\xi > 1$, and

  $$u = \xi \qquad \text{on} \qquad x = \xi t + \xi$$

  for $0 < \xi < 1$. If the shock is located at $x = s(t)$, it follows that

  $$u(x,t) = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x > s(t), \\ x/(1+t) & \text{if } 0 < x < s(t). \end{cases}$$

- The solution ahead of the shock is $u_+ = 0$ and the solution behind the shock is

  $$u_- = \left.\frac{x}{1+t}\right|_{x=s} = \frac{s}{1+t}.$$

  The jump condition across the shock implies that the shock-speed is given by

  $$\frac{ds}{dt} = \frac{1}{2}(u_+ + u_-) = \frac{s}{2(1+t)}.$$

  We also have $s(0) = 1$ for the location of the shock in the initial data.

- Separating variables in this ODE, we get

$$\int \frac{ds}{s} = \int \frac{dt}{2(1+t)}.$$

Evaluating the integrals and using the IC for s, we find that

$$s(t) = \sqrt{1+t},$$

which completes the solution of the IVP.

- The shock strength $[u] = u_- - u_+$ is

$$[u] = \frac{s(t)}{1+t} = \frac{1}{\sqrt{1+t}}.$$

Thus, $[u] \to 0$ as $t \to \infty$.

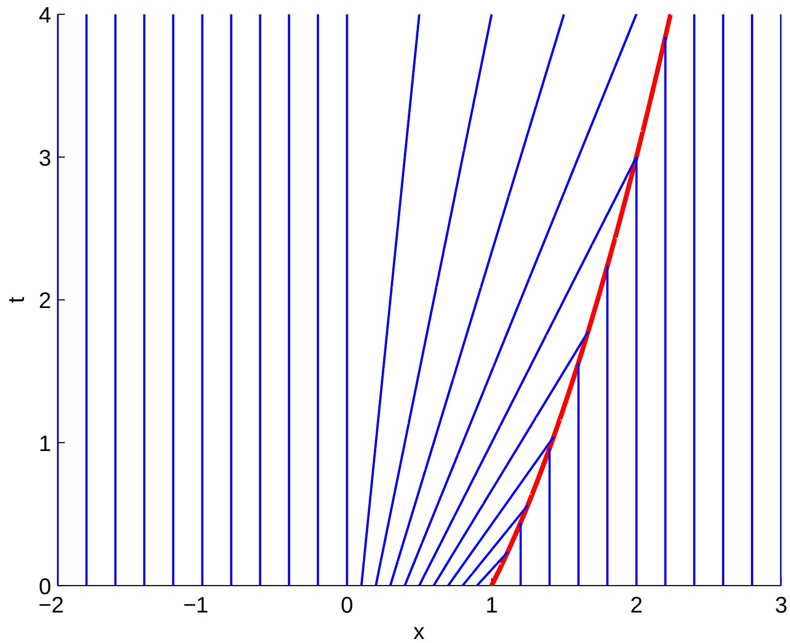

Figure 2: The characteristics are shown in blue and the shock location is in red.

Remark. The shock gets weaker as the expansion wave behind it catches up with it. This nonlinear $t^{-1/2}$-decay in the shock-strength is typical of shocks in localized waves in one-space dimension. In several space dimensions, shocks (e.g., the shocks in a sonic boom generated by a supersonic aircraft) decay even faster as they propagate away from the source due to linear geometrical effects.