Midterm 1: Sample questions
Math 118B, Winter 2014

1. State Green's first and second identities. If Ω is a bounded set with smooth boundary and $\alpha > 0$, use Green's first identity to show that solutions of Poisson's equation with Robin boundary conditions,

$$\begin{aligned} -\Delta u = f \qquad & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \alpha u = g \qquad & \text{on } \partial\Omega, \end{aligned}$$

are unique.

2. Find the Green's function for the BVP

$$\begin{aligned} -u'' = f(x) \qquad & 0 < x < 1, \\ u(0) = A, \qquad & u'(1) = B. \end{aligned}$$

Write down the Green's function representation of the solution.

3. Suppose that $u(\vec{x})$ is the steady temperature distribution of a body, whose heat energy density is proportional to temperature, and $\vec{q}(\vec{x})$ the heat flux vector. If there are no internal heat sources and $\vec{q} = -A\nabla u$ where A is a symmetric matrix, write down the integral form of conservation of energy and derive a PDE for u.

Remark. This constitutive relative for the flux describes anisotropic materials, in which case the heat flux needn't be in the same direction as the temperature gradient.

4. Let $G(\vec{x})$ be the free-space Green's function for the Helmholtz equation

$$-\Delta G + G = \delta(\vec{x}), \qquad G(\vec{x}) \to 0 \quad \text{as } |\vec{x}| \to \infty$$

in three space dimensions. Write down the conditions that determine G, and solve for G. Write down the Green's function representation of the solution of

$$-\Delta u + u = f(\vec{x}), \qquad u(\vec{x}) \to 0 \quad \text{as } |\vec{x}| \to \infty$$

where $f(\vec{x})$ is a smooth function that is zero when $|\vec{x}|$ is sufficiently large.

Hint. The three-dimensional Laplacian of functions $u(r)$ of $r = |\vec{x}|$ is given by

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right).$$

Write $G = H/r$ and solve for H.