Partial Differential Equations
Math 118B, Winter 2014
Solutions: Midterm 1

1. [15%] Let

$$B_\epsilon(0) = \left\{\vec{x} \in \mathbb{R}^3 : |\vec{x}| < \epsilon\right\}, \qquad \partial B_\epsilon(0) = \left\{\vec{x} \in \mathbb{R}^3 : |\vec{x}| = \epsilon\right\}$$

denote the ball and sphere of radius $\epsilon > 0$, respectively, in three space dimensions. Evaluate each of the following limits and say if they are 0, finite and nonzero, or ∞:

$$\text{(a)}\ \lim_{\epsilon\to 0^+} \int_{B_\epsilon(0)} \frac{e^{|\vec{x}|}}{|\vec{x}|^2}\, dV; \quad \text{(b)}\ \lim_{\epsilon\to 0^+} \int_{\partial B_\epsilon(0)} \frac{e^{|\vec{x}|}}{|\vec{x}|^2} dS; \quad \text{(c)}\ \lim_{\epsilon\to 0^+} \int_{\partial B_\epsilon(0)} \log|\vec{x}|\, dS.$$

Solution.

- (a) Using the fact that $dV = 4\pi r^2 dr$ for integrals of spherically symmetric functions in $\mathbb{R}^3$, we get

$$\begin{aligned}\lim_{\epsilon\to 0^+} \int_{B_\epsilon(0)} \frac{e^{|\vec{x}|}}{|\vec{x}|^2}\, dV &= \lim_{\epsilon\to 0^+} 4\pi \int_0^\epsilon \frac{e^r}{r^2}\, r^2\, dr \\ &= \lim_{\epsilon\to 0^+} 4\pi\left(1 - e^\epsilon\right) \\ &= 0.\end{aligned}$$

- (b) Using the fact that the area of the sphere of radius ϵ is $4\pi\epsilon^2$, we get

$$\begin{aligned}\lim_{\epsilon\to 0^+} \int_{\partial B_\epsilon(0)} \frac{e^{|\vec{x}|}}{|\vec{x}|^2} dS &= \lim_{\epsilon\to 0^+} 4\pi\epsilon^2 \cdot \frac{e^\epsilon}{\epsilon^2} \\ &= 4\pi.\end{aligned}$$

  (c) Similarly,

$$\begin{aligned}\lim_{\epsilon\to 0^+} \int_{\partial B_\epsilon(0)} \log|\vec{x}|\, dS &= \lim_{\epsilon\to 0^+} 4\pi\epsilon^2 \cdot \log\epsilon \\ &= 0.\end{aligned}$$

2. [30%] (a) Show that

$$G_F(x) = -\frac{1}{2}|x|$$

is a solution of

$$-\frac{d^2 G_F}{dx^2} = \delta(x), \qquad -\infty < x < \infty.$$

(b) Find the Green's function $G(x;\xi)$ on the interval $0 \le x \le 1$ such that

$$-\frac{d^2 G}{dx^2} = \delta(x-\xi), \qquad 0 < x < 1$$
$$G(0;\xi) = 0, \qquad G(1;\xi) = 0.$$

(c) Write your solution in (b) as

$$G(x;\xi) = G_F(x-\xi) + \phi(x;\xi).$$

Give an explicit expression for ϕ and show that it is a solution of the homogeneous ODE

$$\frac{d^2\phi}{dx^2} = 0.$$

Solution.

- (a) We have

$$G_F(x) = \begin{cases} x/2 & \text{if } x < 0 \\ -x/2 & \text{if } x > 0 \end{cases}$$

  so $d^2G_F/dx^2 = 0$ if $x \neq 0$. Also, G_F is continuous at $x = 0$ and

$$\left[\frac{dG_F}{dx}\right]_{x=0} = \frac{dG_F}{dx}(0^+) - \frac{dG_F}{dx}(0^-) = -\frac{1}{2} - \frac{1}{2} = -1,$$

  so $d^2G_F/d^2x = -\delta(x)$.

- (b) As in the solution for Problem 2 of Homework 2, we get

$$G(x;\xi) = \begin{cases} (1-\xi)x & \text{if } 0 \le x \le \xi, \\ \xi(1-x) & \text{if } \xi \le x \le 1. \end{cases}$$

- (c) We find that

$$
\begin{aligned}
\phi(x;\xi) &= G(x;\xi) - G_F(x) \\
&= \begin{cases} (1-\xi)x - (x-\xi)/2 & \text{if } 0 \le x \le \xi, \\ \xi(1-x) + (x-\xi)/2 & \text{if } \xi \le x \le 1. \end{cases} \\
&= \left(\frac{1}{2} - \xi\right) x + \frac{1}{2}\xi,
\end{aligned}
$$

  so ϕ is a linear function of x, which satisfies the homogeneous equation.

Remark. This problem illustrates the general result that the Green's function of a BVP is equal to the free-space Green's function plus a solution of the homogeneous equation that corrects for the fact that the free-space Green's function doesn't satisfy the boundary conditions.

3. [25%] Suppose a pollutant with concentration $c(\vec{x}, t)$ per unit volume is advected (without diffusion or sources) by a fluid with velocity $\vec{V}(\vec{x}, t)$.

(a) Write down: (i) the rate of change with respect to time of the total amount of pollutant in an arbitrary volume Ω; (ii) the flux of pollutant out of Ω. Give an integral form of conservation of pollutant.

(b) Derive a differential equation for conservation of pollutant from the integral form in (a).

Solution.

- (a) We have

$$\text{rate of change of pollutant in } \Omega = \frac{d}{dt}\int_{\Omega} c(\vec{x}, t)\, d\vec{x},$$
$$\text{flux of pollutant out of } \Omega = \int_{\partial\Omega} c(\vec{x}, t)\vec{V} \cdot \vec{n}\, dS.$$

- Conservation of pollutant implies that

$$\frac{d}{dt}\int_{\Omega} c(\vec{x}, t)\, d\vec{x}, = -\int_{\partial\Omega} c(\vec{x}, t)\vec{V} \cdot \vec{n}\, dS.$$

- (b) Bringing the time derivative inside the integral and using the divergence theorem to rewrite the surface integral as a volume integral, we get

$$\int_{\Omega} \left\{ c_t + \operatorname{div}(c\vec{V}) \right\} d\vec{x} = 0.$$

- Since Ω is arbitrary, it follows that

$$c_t + \operatorname{div}(c\vec{V}) = 0.$$

(We assume these derivatives are continuous.)

4. [30%] Consider the Dirichlet problem for $u(x, y)$ for the two-dimensional Laplacian $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in the upper-half space $y > 0$:

$$\begin{aligned} \Delta u = 0 \qquad & \text{in } y > 0, \\ u(x, 0) = h(x). & \end{aligned} \tag{1}$$

Here, $h(x)$ is a continuous function which is zero when $|x|$ is sufficiently large.

(a) If $\vec{x} = (x, y)$ and $\vec{\xi} = (\xi, \eta)$ with $\eta > 0$, show that the Green's function for this problem corresponding to a point source at $\vec{\xi}$ is

$$G(\vec{x}; \vec{\xi}) = -\frac{1}{2\pi} \log \left| \vec{x} - \vec{\xi} \right| + \frac{1}{2\pi} \log \left| \vec{x} - \vec{\xi}^* \right|$$

where $\vec{\xi}^* = (\xi, -\eta)$.

Hint. You can assume, without proof, that $G_F(\vec{x}) = -(1/2\pi) \log |\vec{x}|$ is the free space Green's function for the Laplacian, such that $-\Delta G_F = \delta(\vec{x})$.

(b) Use Green's second identity and a formal calculation with δ-functions to derive an integral representation of the solution u of (1).

Solution.

- (a) We have

  $$-\Delta G = \delta(\vec{x} - \vec{\xi}) - \delta(\vec{x} - \vec{\xi}^*) = \delta(\vec{x} - \vec{\xi}) \qquad \text{in } y > 0,$$

  since $\vec{x} \neq \vec{\xi}^*$ if $y > 0$.

- We have

  $$|\vec{x} - \vec{\xi}| = \sqrt{(x - \xi)^2 + (y - \eta)^2}, \qquad |\vec{x} - \vec{\xi}^*| = \sqrt{(x - \xi)^2 + (y + \eta)^2},$$

  so $|\vec{x} - \vec{\xi}| = |\vec{x} - \vec{\xi}^*|$ when $y = 0$, which implies that $G = 0$ on $y = 0$. This shows that $G(\vec{x}; \vec{\xi})$ is the Green's function for (1). (Note also that $G \to 0$ as $|\vec{x}| \to \infty$.)

- (b) Let

  $$\Omega = \{(x, y) : -\infty < x < \infty, y > 0\}, \quad \partial\Omega = \{(x, 0) : -\infty < x < \infty\}$$

denote the upper-half plane and the x-axis. From Green's second identity,

$$\int_{\Omega} \left\{ G(\vec{x};\vec{\xi})\Delta u(\vec{x}) - u(\vec{x})\Delta G(\vec{x};\vec{\xi}) \right\} d\vec{x} = \int_{\partial\Omega} \left\{ G(\vec{x};\vec{\xi})\frac{\partial u}{\partial n}(\vec{x}) - u(\vec{x})\frac{\partial G}{\partial n}(\vec{x};\vec{\xi}) \right\} ds.$$

(We assume that the functions decay sufficiently rapidly for any boundary terms to vanish as $|\vec{x}| \to \infty$.)

- Using the equations satisfied by u and G, we get

$$\int_{\Omega} u(\vec{x})\delta(\vec{x}-\vec{\xi})\, d\vec{x} = -\int_{\partial\Omega} u(\vec{x})\frac{\partial G}{\partial n}(\vec{x};\vec{\xi})\, ds.$$

The outward normal derivative to Ω on $\partial\Omega$ is $\partial/\partial n = -\partial/\partial y$, and $ds = dx$, so

$$u(\vec{\xi}) = \int_{-\infty}^{\infty} h(x) \left. \frac{\partial G}{\partial y}(\vec{x};\vec{\xi}) \right|_{y=0} dx.$$

- Differentiating G, we find that

$$\frac{\partial G}{\partial y}(\vec{x};\vec{\xi}) = -\frac{1}{2\pi}\frac{y-\eta}{|\vec{x}-\vec{\xi}|} + \frac{1}{2\pi}\frac{y+\eta}{|\vec{x}-\vec{\xi}^{*}|},$$

and

$$\left. \frac{\partial G}{\partial y}(\vec{x};\vec{\xi}) \right|_{y=0} = \frac{1}{\pi}\frac{\eta}{|\vec{x}-\vec{\xi}|}.$$

- The solution for u is therefore

$$u(\xi,\eta) = \frac{\eta}{\pi}\int_{-\infty}^{\infty} \frac{h(x)}{\sqrt{(x-\xi)^2+\eta^2}}\, dx.$$

Remark. The Green's function here is the one given by the method of images, and the final solution for u is called Poisson's formula for the half-plane. It's closely related to Poisson's formula for a disc that we derived last quarter by separation of variables; the formula for a disc can also be derived by the method of images.