Solutions: Midterm 2
Math 118B, Winter 2014

1. [25%] (a) Let $a > 0$. Compute the inverse Fourier transform $s(x)$ of the function

$$S(k) = \begin{cases} 1 & \text{if } -a < k < a, \\ 0 & \text{if } |k| > a. \end{cases}$$

(b) Suppose that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk$$

has Fourier transform $F(k)$ and $f_a(x)$ is the "bandlimited" function

$$f_a(x) = \frac{1}{2\pi} \int_{-a}^{a} F(k) e^{ikx}\, dk.$$

Express f_a in terms of a convolution involving f.

Solution.

- (a) If $x \neq 0$, then

  $$\begin{aligned} s(x) &= \frac{1}{2\pi} \int_{-a}^{a} e^{ikx}\, dk \\ &= \frac{1}{2\pi x} \left(e^{iax} - e^{-iax} \right) \\ &= \frac{\sin ax}{\pi x}, \end{aligned}$$

  and if $x = 0$, then

  $$\begin{aligned} s(0) &= \frac{1}{2\pi} \int_{-a}^{a} dk \\ &= \frac{a}{\pi}. \end{aligned}$$

- Thus, $s(x) = (a/\pi)\operatorname{sinc}(ax)$ where

  $$\operatorname{sinc} x = \begin{cases} (\sin x)/x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

  is the sinc-function. (See the graph on the next page.)

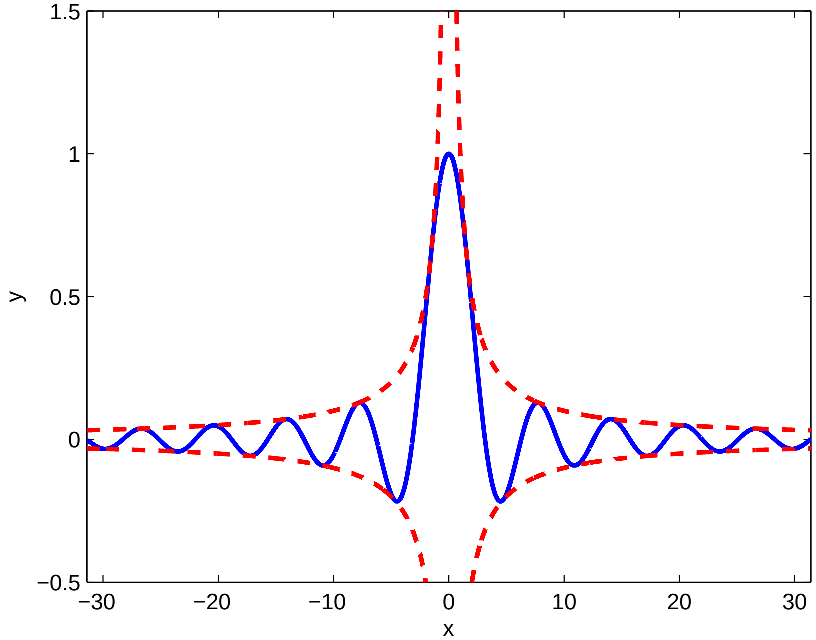

Figure 1: Graph of $y = \operatorname{sinc} x$. The dashed lines are $y = \pm 1/x$.

- (b) The Fourier transform $F_a = \mathcal{F}[f_a]$ is

$$F_a(k) = F(k)S(k),$$

so the convolution theorem implies that $f_a = f * s$, or

$$f_a(x) = \int_{-\infty}^{\infty} f(x-y) \frac{\sin ay}{\pi y} \, dy.$$

Remark. Because the sinc-function has such slowly decaying oscillations, the sharp band-limiting of a function may produce spurious oscillations. These "ringing artifacts" are often undesirable in signal processing.

2. [25%] (a) Give the definition of the δ-function as a distribution on $\mathbb{R}$.
(b) Define the derivative T' of a distribution T on $\mathbb{R}$.
(c) Verify from the definitions that

$$T = \frac{1}{2}e^{-|x|}$$

is a distributional solution of the ODE

$$-T'' + T = \delta.$$

Solution.

- (a) The δ-function is the linear map $\delta : \mathcal{D} \to \mathbb{R}$ on the space $\mathcal{D}$ of test functions $\phi(x)$ defined by $\langle \delta, \phi \rangle = \phi(0)$.
- (b) If $T : \mathcal{D} \to \mathbb{R}$ is a distribution, then its derivative $T' : \mathcal{D} \to \mathbb{R}$ is defined by $\langle T', \phi \rangle = -\langle T, \phi' \rangle$.
- (c) Using the definition $\langle T'', \phi \rangle = \langle T, \phi'' \rangle$ of the distributional derivative and integrating by parts twice to simplify the result, we find that

$$\begin{aligned}
\langle T'', \phi \rangle &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x|}\phi''(x)\,dx \\
&= \frac{1}{2}\int_{-\infty}^{0} e^{x}\phi''(x)\,dx + \frac{1}{2}\int_{0}^{\infty} e^{-x}\phi''(x)\,dx \\
&= \frac{1}{2}\left[e^{x}\phi'(x)\right]_{-\infty}^{0} - \frac{1}{2}\int_{-\infty}^{0} e^{x}\phi'(x)\,dx \\
&\quad + \frac{1}{2}\left[e^{-x}\phi'(x)\right]_{0}^{\infty} + \frac{1}{2}\int_{0}^{\infty} e^{-x}\phi'(x)\,dx \\
&= -\frac{1}{2}\left[e^{x}\phi(x)\right]_{-\infty}^{0} + \frac{1}{2}\int_{-\infty}^{0} e^{x}\phi(x)\,dx \\
&\quad + \frac{1}{2}\left[e^{-x}\phi(x)\right]_{0}^{\infty} + \frac{1}{2}\int_{0}^{\infty} e^{-x}\phi(x)\,dx \\
&= -\phi(0) + \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x|}\phi(x)\,dx \\
&= \langle -\delta + T, \phi \rangle.
\end{aligned}$$

  It follows that $T'' = T - \delta$, which proves the result.

3. [25%] (a) Define what it means for a sequence of distributions T_n to converge weakly to a distribution T as $n \to \infty$.
(b) Let

$$f_n(x) = \begin{cases} n/2 & \text{if } -1/n < x < 1/n, \\ 0 & \text{if } |x| > 1/n. \end{cases}$$

Show that $f_n \to \delta$ weakly as $n \to \infty$.

Solution.

- (a) We have $T_n \to T$ weakly if $\langle T_n, \phi \rangle \to \langle T, \phi \rangle$ in $\mathbb{R}$ for every test function ϕ.
- (b) As $n \to \infty$, we have

  $$\begin{aligned} \langle f_n, \phi \rangle &= \frac{n}{2} \int_{-1/n}^{1/n} \phi(x)\, dx \\ &= \text{average value of } \phi(x) \text{ on } -1/n \le x \le 1/n \\ &\to \phi(0) = \langle \delta, \phi \rangle. \end{aligned}$$

  It follows that $f_n \to \delta$ weakly as $n \to \infty$.

Remark. Here's a proof that the average values of a continuous function converge to the value of the function. Suppose that $\phi(x)$ is continuous at $x = 0$. Subtracting and adding $\phi(0)$ inside the integral, we can write

$$\frac{n}{2} \int_{-1/n}^{1/n} \phi(x)\, dx = \frac{n}{2} \int_{-1/n}^{1/n} [\phi(x) - \phi(0)]\, dx + \phi(0).$$

Given any $\epsilon > 0$, there exists $\delta > 0$ such that $|\phi(x) - \phi(0)| < \epsilon$ when $|x| < \delta$. (This is the definition of the continuity of $\phi(x)$ at 0.) Choose a positive integer $N \in \mathbb{N}$ such that $1/N < \delta$. If $n > N$, then $|x| < \delta$ if $|x| \le 1/n$, so

$$\begin{aligned} \left| \frac{n}{2} \int_{-1/n}^{1/n} \phi(x)\, dx - \phi(0) \right| &= \left| \frac{n}{2} \int_{-1/n}^{1/n} [\phi(x) - \phi(0)]\, dx \right| \\ &\le \frac{n}{2} \int_{-1/n}^{1/n} |\phi(x) - \phi(0)|\, dx \\ &< \frac{n}{2} \int_{-1/n}^{1/n} \epsilon\, dx = \epsilon. \end{aligned}$$

This proves that $\frac{n}{2} \int_{-1/n}^{1/n} \phi(x)\, dx \to \phi(0)$ as $n \to \infty$.

4. [25%] Use the Fourier transform to find the solution $u(x,t)$ of the following initial value problem:

$$u_t + u_{xx} + u_{xxxx} = 0 \qquad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = f(x).$$

You should write your answer as a Fourier integral, but you don't need to invert the transform. How do you expect the solution to behave as $t \to +\infty$?

Solution.

- Taking the Fourier transform with respect to x of the PDE, with

$$U(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}\,dx,$$

 when x-derivatives transform to multiplication by ik, we get

$$U_t + (-k^2 + k^4)U = 0, \qquad U(k,0) = F(k),$$

 where $F = \mathcal{F}[f]$ is the Fourier transform of f.

- The solution of this ODE is $U(k,t) = F(k)e^{(k^2-k^4)t}$, and the solution for $u(x,t)$ is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{(k^2-k^4)t}e^{ikx}\,dk.$$

- By the convolution theorem, the solution can also be written as

$$u(x,t) = \int_{-\infty}^{\infty} f(x-y)G(y,t)\,dy,$$

 where the Green's function G is given by

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+(k^2-k^4)t}\,dk.$$

- The Fourier modes with $|k| < 1$ grow exponentially in time, so the solution will grow exponentially in time (unless $F(k) = 0$ for $|k| < 1$).

Remark. We can derive a more precise description of the long-time behavior of the solution as follows. The most unstable (positive) wavenumber is $k = k_0$ where $k_0 = 1/\sqrt{2}$, at which the growth rate $\sigma(k) = k^2 - k^4$ attains its maximum value of $\sigma(k_0) = 1/4$ and $\sigma'(k_0) = 0$. Suppose, for definiteness, that $F(k)$ is a smooth, rapidly decaying function (e.g. a Schwartz function) and $F(k_0) \neq 0$, meaning that the initial data contains non-zero, maximally unstable modes. Since the initial data is real, $F(-k_0) = F^*(k_0)$.

For large times t, the dominant contribution to the Fourier integral for u comes from values of k close to $\pm k_0$. To leading order, we can approximate the contribution from $k = k_0$ by evaluating $F(k)$ at $k = k_0$ and Taylor expanding the growth rate $\sigma(k)$ about $k = k_0$ up to the quadratic term,

$$\sigma(k) = \sigma(k_0) + \frac{1}{2}\sigma''(k_0)(k-k_0)^2 + O(k-k_0)^3 = \frac{1}{4} - 2(k-k_0)^2 + O(k-k_0)^3.$$

Furthermore, we can integrate the resulting approximations over all k, since the contributions from values of k that are not close to k_0 are negligible. A similar approximation near $k = -k_0$ gives the complex conjugate of the contribution from $k = k_0$.

This procedure gives the large-t asymptotic approximation

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k) e^{(k^2-k^4)t} e^{ikx}\, dk \\ &\sim \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k_0) e^{(1/4-2(k-k_0)^2)t} e^{ik_0 x}\, dk + \text{c.c.} \\ &\sim \frac{1}{2\pi} F(k_0) e^{ik_0 x} e^{t/4} \int_{-\infty}^{\infty} e^{-2(k-k_0)^2 t}\, dk + \text{c.c.}, \end{aligned}$$

where c.c. stands for the complex conjugate of the preceding term. Using the standard Gaussian integral, we get

$$\int_{-\infty}^{\infty} e^{-2(k-k_0)^2 t}\, dk = \frac{1}{\sqrt{2t}}\int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = \sqrt{\frac{\pi}{2t}}.$$

Writing $F(k_0) = ae^{i\delta}$, where $a = |F(k_0)|$ and $\delta = \arg F(k_0)$, we get that

$$\begin{aligned} u(x,t) &\sim \frac{e^{t/4}}{2\sqrt{2\pi t}} F(k_0) e^{ik_0 x} + \text{c.c.} \\ &\sim \frac{ae^{t/4}}{\sqrt{2\pi t}} \cos\left(\frac{x}{\sqrt{2}} + \delta\right) \qquad \text{as } t \to +\infty. \end{aligned}$$