Problem Set 1: Math 118B

Winter Quarter, 2014

1. Suppose that C is a positively oriented, simple closed curve in the (x, y)-plane $z = 0$ that encloses an area A. Let $\vec{u} = u(x, y)\vec{i} + v(x, y)\vec{j}$ denote a vector field in the (x, y)-plane.

(a) Compute $\operatorname{curl}\vec{u}$ and show that it is in the $\vec{k}$ direction. Use Stokes theorem to derive the planar version of Green's theorem

$$\int_A (v_x - u_y)\; dxdy = \int_C (u\, dx + v\, dy)\,.$$

(b) Verify this identity explicitly for $\vec{u} = -y\vec{i} + x\vec{j}$ when C is the circle of radius a with parametric equation $x = a\cos t$, $y = a\sin t$.

2. Define a scalar field ϕ in $\mathbb{R}^2$ or $\mathbb{R}^3$ by

$$\phi(x, y) = \log\left(x^2 + y^2\right), \qquad \phi(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

In each case, compute $\nabla\phi$ and show that $\Delta\phi = 0$ for $\vec{x} \neq 0$.

3. (a) If ϕ is a scalar field and the curve C is the boundary of a surface S, show that

$$\int_C \nabla\phi \cdot d\vec{x} = 0.$$

(b) If $\vec{u}$ is a vector field and the surface $\partial\Omega$ is the boundary of a volume Ω, show that

$$\int_{\partial\Omega} \operatorname{curl}\vec{u}\, dS = 0.$$

4. We use subscript notation and write $\vec{x} = (x_1, x_2, x_3)$, $\vec{n} = (n_1, n_2, n_3)$.

If u, v are scalar fields and Ω is a volume in $\mathbb{R}^3$ with boundary $\partial\Omega$ and outward normal $\vec{n}$, use the divergence theorem to show that

$$\int_\Omega u\frac{\partial v}{\partial x_i}\, dV = \int_{\partial\Omega} uvn_i\, dS - \int_\Omega v\frac{\partial u}{\partial x_i}\, dV.$$

(This result shows that the divergence theorem can be regarded as a multi-dimensional version of integration by parts.)

5. Suppose a fluid flowing in $\mathbb{R}^3$ has mass-density $\rho(\vec{x},t)$ and velocity $\vec{u}(\vec{x},t)$.

(a) Let $\Omega \subset \mathbb{R}^3$ be an arbitrary volume. Explain why conservation of mass implies that

$$\frac{d}{dt}\int_{\Omega} \rho \, dV = -\int_{\partial\Omega} \rho\vec{u}\cdot\vec{n}\, dS.$$

(b) If ρ, $\vec{u}$ are smooth functions, deduce that they satisfy the differential form of conservation of mass

$$\rho_t + \nabla\cdot(\rho\vec{u}) = 0.$$

6. Maxwell's equations for time-dependent electric and magnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ in a vacuum are

$$\begin{aligned} &\vec{E}_t - c^2 \operatorname{curl}\vec{B} = 0,\\ &\vec{B}_t + \operatorname{curl}\vec{E} = 0,\\ &\operatorname{div}\vec{E} = 0,\\ &\operatorname{div}\vec{B} = 0, \end{aligned}$$

where c is a constant. Show that

$$\vec{E}_{tt} = c^2\Delta\vec{E}.$$

What is the interpretation of c?

Hint. You can assume the vector identity

$$\operatorname{curl}\left(\operatorname{curl}\vec{E}\right) = \nabla\left(\operatorname{div}\vec{E}\right) - \Delta\vec{E},$$

where the Laplacian of a vector field is defined component-wise in Cartesian coordinates i.e., $\Delta\left(E\vec{i} + F\vec{j} + H\vec{k}\right) = (\Delta E)\vec{i} + (\Delta F)\vec{j} + (\Delta H)\vec{k}$.