Problem Set 1: Solutions
Math 118B: Winter Quarter, 2014

1. Suppose that C is a positively oriented, simple closed curve in the (x, y)-plane $z = 0$ that encloses an area A. Let $\vec{u} = u(x, y)\vec{i} + v(x, y)\vec{j}$ denote a vector field in the (x, y)-plane.
(a) Compute $\operatorname{curl}\vec{u}$ and show that it is in the $\vec{k}$ direction. Use Stokes theorem to derive the planar version of Green's theorem

$$\int_A (v_x - u_y)\, dxdy = \int_C (u\, dx + v\, dy)\,. \tag{1}$$

(b) Verify this identity explicitly for $\vec{u} = -y\vec{i} + x\vec{j}$ when C is the circle of radius a with parametric equation $x = a\cos t$, $y = a\sin t$.

Solution.

- (a) We have
$$\begin{aligned}\operatorname{curl}\vec{u} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u(x,y) & v(x,y) & 0 \end{vmatrix} \\ &= (v_x - u_y)\vec{k}.\end{aligned}$$

- If C is positively oriented in the (x, y)-plane, then the corresponding normal to the area A enclosed by $C = \partial A$ is $\vec{n} = \vec{k}$. It follows that $\operatorname{curl}\vec{u}\cdot\vec{n} = (v_x - u_y)$ and $\vec{u}\cdot d\vec{x} = u\, dx + v\, dy$, so Stokes theorem
$$\int_A \operatorname{curl}\vec{u}\cdot\vec{n}\, dS = \int_{\partial A} \vec{u}\cdot d\vec{x}$$
gives Green's theorem (1).

- (b) In this case, we have
$$\begin{aligned}\int_A (v_x - u_y)\, dxdy &= \int_A 2\, dxdy = 2\pi a^2, \\ \int_C (u\, dx + v\, dy) &= \int_C -y\, dx + x\, dy \\ &= \int_0^{2\pi} \left\{(-a\sin t)\cdot(-a\cos t) + (a\cos t)\cdot(a\sin t)\right\}\, dt \\ &= \int_0^{2\pi} a^2\, dt = 2\pi a^2.\end{aligned}$$
which verifies the identity.

2. Define a scalar field ϕ in $\mathbb{R}^2$ or $\mathbb{R}^3$ by

$$\phi(x,y) = \log\left(x^2+y^2\right), \qquad \phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}.$$

In each case, compute $\nabla\phi$ and show that $\Delta\phi = 0$ for $\vec{x} \neq 0$.

Solution.

- (a) In each case, we have

$$\nabla\phi = (\phi_x, \phi_y) = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right),$$
$$\nabla\phi = (\phi_x, \phi_y, \phi_z) = -\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right),$$

and

$$\begin{aligned}
\Delta\phi &= \phi_{xx} + \phi_y \\
&= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \\
&= 0, \\
\Delta\phi &= \phi_{xx} + \phi_{yy} + \phi_{zz} \\
&= \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{x^2}{(x^2+y^2+z^2)^{3/2}} \\
&\quad + \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{y^2}{(x^2+y^2+z^2)^{3/2}} \\
&\quad + \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{z^2}{(x^2+y^2+z^2)^{3/2}} \\
&= 0.
\end{aligned}$$

3. (a) If ϕ is a scalar field and the curve C is the boundary of a surface S, show that

$$\int_C \nabla\phi \cdot d\vec{x} = 0.$$

(b) If $\vec{u}$ is a vector field and the surface $\partial\Omega$ is the boundary of a volume Ω, show that

$$\int_{\partial\Omega} \operatorname{curl} \vec{u}\, dS = 0.$$

Solution.

- By Stokes' theorem,

$$\int_C \nabla\phi \cdot d\vec{x} = \int_S \operatorname{curl} \nabla\phi\, dS = 0$$

  since the curl of a gradient is zero.

- Alternatively, note that if C is a curve from P to Q then

$$\int_C \nabla\phi \cdot d\vec{x} = \phi(Q) - \phi(P),$$

  and this is zero since the bounding curve of a surface is closed, so $P = Q$.

- By the divergence theorem

$$\int_{\partial\Omega} \operatorname{curl} \vec{u}\, dS = \int_\Omega \operatorname{div} \operatorname{curl} \vec{u}\, dV = 0$$

  since the divergence of a curl is zero.

4. We use subscript notation and write $\vec{x} = (x_1, x_2, x_3)$, $\vec{n} = (n_1, n_2, n_3)$. If u, v are scalar fields and Ω is a volume in $\mathbb{R}^3$ with boundary $\partial\Omega$ and outward normal $\vec{n}$, use the divergence theorem to show that

$$\int_\Omega u \frac{\partial v}{\partial x_i}\, dV = \int_{\partial\Omega} uvn_i\, dS - \int_\Omega v \frac{\partial u}{\partial x_i}\, dV.$$

(This result shows that the divergence theorem can be regarded as a multi-dimensional version of integration by parts.)

Solution.

- Let $\vec{w} = uv\vec{e}_i$, where $\vec{e}_i$ is the ith coordinate vector i.e., the ith component of $\vec{w}$ is uv and all its other components are zero. Then

  $$\operatorname{div} \vec{w} = \frac{\partial}{\partial x_i}(uv) = u\frac{\partial v}{\partial x_i} + v\frac{\partial u}{\partial x_i}.$$

  and $\vec{w} \cdot \vec{n} = uvn_i$ where n_i is the ith component of $\vec{n}$.

- Then the divergence theorem

  $$\int_\Omega \operatorname{div} \vec{w}\, dV = \int_{\partial\Omega} \vec{w} \cdot \vec{n}\, dS$$

  implies that

  $$\int_\Omega \left(u\frac{\partial v}{\partial x_i} + v\frac{\partial u}{\partial x_i} \right) dV = \int_{\partial\Omega} uvn_i\, dS,$$

  which proves the result.

5. Suppose a fluid flowing in $\mathbb{R}^3$ has mass-density $\rho(\vec{x}, t)$ and velocity $\vec{u}(\vec{x}, t)$.
(a) Let $\Omega \subset \mathbb{R}^3$ be an arbitrary volume. Explain why conservation of mass implies that

$$\frac{d}{dt}\int_\Omega \rho\, dV = -\int_{\partial\Omega} \rho\vec{u}\cdot\vec{n}\, dS.$$

(b) If ρ, $\vec{u}$ are smooth functions, deduce that they satisfy the differential form of conservation of mass

$$\rho_t + \operatorname{div}(\rho\vec{u}) = 0.$$

Solution.

- (a) This equation says that the rate of change of the total mass of fluid inside Ω is equal to the rate at which mass flows out of Ω through its boundary.

- (b) Since Ω is a fixed spatial volume that does not change in time, we can bring the time derivative inside the integral:

$$\frac{d}{dt}\int_\Omega \rho\, dV = \int_\Omega \rho_t\, dV.$$

- We can use the divergence theorem to rewrite the surface integral as a volume integral:

$$\int_{\partial\Omega} \rho\vec{u}\cdot\vec{n}\, dS = \int_\Omega \operatorname{div}(\rho\vec{u})\, dV.$$

- It follows that

$$\int_\Omega \{\rho_t + \operatorname{div}(\rho\vec{u})\}\ dV = 0$$

  for an arbitrary volume Ω, which implies that the integrand is identically zero (assuming that it is a continuous function).

6. Maxwell's equations for time-dependent electric and magnetic fields $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$ in a vacuum are

$$\begin{aligned}
&\vec{E}_t - c^2 \operatorname{curl} \vec{B} = 0, \\
&\vec{B}_t + \operatorname{curl} \vec{E} = 0, \\
&\operatorname{div} \vec{E} = 0, \\
&\operatorname{div} \vec{B} = 0,
\end{aligned}$$

where c is a constant. Show that

$$\vec{E}_{tt} = c^2 \Delta \vec{E}.$$

What is the interpretation of c?

Hint. You can assume the vector identity

$$\operatorname{curl}\left(\operatorname{curl} \vec{E}\right) = \nabla\left(\operatorname{div} \vec{E}\right) - \Delta \vec{E},$$

where the Laplacian of a vector field is defined component-wise in Cartesian coordinates i.e., $\Delta\left(E\vec{i} + F\vec{j} + H\vec{k}\right) = (\Delta E)\vec{i} + (\Delta F)\vec{j} + (\Delta H)\vec{k}$.

Solution.

- Taking the time derivative of the first equation and using the second equation to eliminate $\vec{B}_t$, we get

  $$\vec{E}_{tt} = c^2 \operatorname{curl} \vec{B}_t = -c^2 \operatorname{curl}\operatorname{curl} \vec{E}.$$

  From the vector identity and the equation $\operatorname{div} \vec{E} = 0$, we have

  $$\operatorname{curl}\operatorname{curl} \vec{E} = -\Delta \vec{E},$$

  and the result follows.

- The constant c is the speed of light.

Remark. Maxwell (1865) introduced the term $\vec{E}_t$ (called the "displacement current") in the first equation, computed the speed c from purely electrostatic and magnetostatic measurements, and used the result to conjecture that light is an electromagnetic phenomenon. This conjecture was later verified experimentally by Hertz (1888).