Problem Set 2: Solutions

Math 118B: Winter Quarter, 2014

1. (a) Find the Green's function $G(x)$ for the ODE

$$-\frac{d^2G}{dx^2} + G = \delta(x) \qquad -\infty < x < \infty$$

where $G(x) \to 0$ as $|x| \to \infty$.

(b) Write down the Green's function representation of the solution $u(x)$ of

$$-\frac{d^2u}{dx^2} + u(x) = f(x) \qquad -\infty < x < \infty$$

where $u(x) \to 0$ as $|x| \to \infty$ and $f(x)$ is a given (smooth) function that is zero outside a bounded set.

(c) Verify explicitly that the your expression for $u(x)$ in (b) is a solution.

(d) Give a physical interpretation of this problem (in terms of heat flow, for example).

Solution.

- (a) For $-\infty < x < 0$,

  $$-\frac{d^2G}{dx^2} + G = 0 \qquad G(x) \to 0 \text{ as } x \to -\infty,$$

  and for $0 < x < \infty$,

  $$-\frac{d^2G}{dx^2} + G = 0 \qquad G(x) \to 0 \text{ as } x \to \infty.$$

  This gives

  $$G(x) = \begin{cases} Ae^{x} & \text{if } -\infty < x < 0, \\ Be^{-x} & \text{if } 0 < x < \infty, \end{cases}$$

  for some constants A, B.

- At $x = 0$, we require that: (i) $G(x)$ is continuous; (ii) the derivative dG/dx jumps by -1. Condition (i) gives $A = B$; and (ii) gives

  $$1 = -\left[\frac{dG}{dx}\right]_{x=0} = -\frac{dG}{dx}(0^+) + \frac{dG}{dx}(0^-) = B + A,$$

  so $A = B = 1/2$. It follows that

  $$G(x) = \frac{1}{2}e^{-|x|}.$$

- (b) The Green's function representation of the solution is
$$u(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\xi|} f(\xi)\, d\xi. \tag{1}$$
- (c) To verify that (1) is the solution, we write it as
$$u(x) = \frac{1}{2}\left(\int_{-\infty}^{x} e^{\xi-x} f(\xi)\, d\xi + \int_{x}^{\infty} e^{x-\xi} f(\xi)\, d\xi\right).$$
Differentiating once, we get
$$\begin{aligned}\frac{du}{dx}(x) &= \frac{1}{2}\left(f(x) - \int_{-\infty}^{x} e^{\xi-x} f(\xi)\, d\xi - f(x) + \int_{x}^{\infty} e^{x-\xi} f(\xi)\, d\xi\right)\\ &= \frac{1}{2}\left(-\int_{-\infty}^{x} e^{\xi-x} f(\xi)\, d\xi + \int_{x}^{\infty} e^{x-\xi} f(\xi)\, d\xi\right)\end{aligned}$$
Differentiating again, we get
$$\begin{aligned}\frac{d^2u}{dx^2}(x) &= \frac{1}{2}\left(-f(x) + \int_{-\infty}^{x} e^{\xi-x} f(\xi)\, d\xi - f(x) + \int_{x}^{\infty} e^{x-\xi} f(\xi)\, d\xi\right)\\ &= -f(x) + \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-\xi|} f(\xi)\, d\xi\\ &= -f(x) + u,\end{aligned}$$
which shows that u is a solution of (1).
- Suppose that $f(x) = 0$ for $|x| > R$. If $x > R$, then
$$u(x) = \frac{1}{2}e^{-x}\int_{-R}^{R} e^{\xi} f(\xi)\, d\xi \to 0 \qquad \text{as } x \to \infty,$$
and if $x < -R$, then
$$u(x) = \frac{1}{2}e^{x}\int_{-R}^{R} e^{-\xi} f(\xi)\, d\xi \to 0 \qquad \text{as } x \to -\infty.$$
This shows that (1) satisfies $u(x) \to 0$ as $|x| \to \infty$.
- (d) The heat equation
$$u_t = u_{xx} - u + f(x)$$
describes the flow of heat in a non-insulated rod with temperature u. The term u_{xx} describes the diffusion of heat, the term $-u$ describes the loss of heat to the surroundings at temperature 0 (Newton's law of cooling), and $f(x)$ is the density of internal heat sources. The ODE describes the resulting steady-state temperature (with $u_t = 0$) in an infinite rod.

2. (a) Find the Green's function $G(x;\xi)$ for the BVP

$$-\frac{d^2G}{dx^2} = \delta(x-\xi), \qquad 0 < x < 1$$
$$G(0;\xi) = 0, \qquad G(1;\xi) = 0,$$

where $0 < \xi < 1$. (Note: In this problem $G(x;\xi)$ isn't a function of $x - \xi$ because of the boundary conditions.)

(b) Sketch the graph of the Green's function $G(x;\xi)$ versus x for a few different values of ξ. Give a physical interpretation of the BVP in terms of: (i) heat flow; (ii) an elastic string. Does the Green's function look the way you would expect?

(c) Use the superposition principle to explain why you expect the solution of the BVP

$$-\frac{d^2u}{dx^2} = f(x), \qquad 0 < x < 1$$
$$u(0) = 0, \qquad u(1) = 0$$

to have the Green's function representation

$$u(x) = \int_0^1 G(x;\xi)f(\xi)\,d\xi.$$

(d) Evaluate the Green's function representation for $u(x)$ in (c) explicitly if $f(x) = \sin\pi x$. Verify that it gives the solution of the BVP.

Solution.

- (a) For $0 < x < \xi$,

  $$-\frac{d^2G}{dx^2} = 0 \qquad G(0;\xi) = 0),$$

  and for $\xi < x < 1$,

  $$-\frac{d^2G}{dx^2} = 0 \qquad G(1;\xi).$$

  This gives

  $$G(x;\xi) = \begin{cases} Ax & \text{if } 0 < x < \xi, \\ B(1-x) & \text{if } \xi < x < 1, \end{cases}$$

  where the constants of integration A, B can depend on the location ξ of the point source.

- At $x = \xi$, we require that: (i) $G(x)$ is continuous; (ii) the derivative dG/dx jumps by -1. Condition (i) gives

$$A = C(1-\xi), \qquad B = C\xi$$

for some constant C; and (ii) gives

$$1 = -\left[\frac{dG}{dx}\right]_{x=\xi} = -\frac{dG}{dx}(\xi^+;\xi) + \frac{dG}{dx}(\xi^-;\xi) = B + A = C,$$

so $C = 1$. It follows that

$$G(x;\xi) = \begin{cases} (1-\xi)x & \text{if } 0 < x < \xi, \\ \xi(1-x) & \text{if } \xi < x < 1. \end{cases}$$

- (b) The Green's function describes: (i) the steady temperature in a laterally insulated rod, whose endpoints are held at 0 temperature, due to a unit point heat source located at ξ; (ii) the equilibrium displacement of an elastic string, whose endpoints are fixed, due to a unit point force applied at ξ.

- (c) The Green's function representation of the solution is the linear superposition of point source solutions with density f:

$$\begin{aligned} u(x) &= \int_0^1 G(x;\xi)f(\xi)\,d\xi \\ &= (1-x)\int_0^x \xi f(\xi)\,d\xi + x\int_x^1 (1-\xi)f(\xi)\,d\xi. \end{aligned} \tag{2}$$

- (d) Using $f(x) = \sin \pi x$ in (2) and integrating by parts in the result, we get

$$\begin{aligned} u(x) &= (1-x)\int_0^x \xi \sin \pi\xi\,d\xi + x\int_x^1 (1-\xi)\sin \pi\xi\,d\xi \\ &= (1-x)\left[-\frac{\xi}{\pi}\cos \pi\xi + \frac{1}{\pi^2}\sin \pi\xi\right]_0^x - x\left[\frac{(1-\xi)}{\pi}\cos \pi\xi + \frac{1}{\pi^2}\sin \pi\xi\right]_x^1 \\ &= (1-x)\left[-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right] + x\left[\frac{(1-x)}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right] \\ &= \frac{1}{\pi^2}\sin \pi x, \end{aligned}$$

which is the correct solution.