Partial Differential Equations

Math 118B, Winter 2018

Final: Solutions

1. [10%] Suppose that $u(r, \theta)$ is harmonic in the open disc of radius 2 and continuous on the closed disc of radius 2, where (r, θ) are polar coordinates. If

$$u(2, \theta) = \theta(2\pi - \theta) \qquad 0 \leq \theta \leq 2\pi,$$

show that $0 \leq u(r, \theta) \leq \pi^2$ for all $0 \leq r \leq 2$.

Solution

- We have

  $$0 \leq \theta(2\pi - \theta) \leq \pi^2 \qquad \text{for } 0 \leq \theta \leq 2\pi,$$

  so the maximum principle for harmonic functions implies that

  $$0 \leq u(r, \theta) \leq \pi^2 \qquad \text{for } 0 \leq r \leq 2.$$

2. [20%] Let $a, b > 0$, and consider the following BVP for $u(x,y)$ on the rectangle $0 \le x \le a$, $0 \le y \le b$:

$$\begin{aligned} -\left(u_{xx} + u_{yy}\right) &= \lambda u, \\ u(0,y) = 0, \quad u(a,y) &= 0 \qquad \text{for } 0 \le y \le b, \\ u(x,0) = 0, \quad u(x,b) &= 0 \qquad \text{for } 0 \le x \le a, \end{aligned}$$

where $\lambda > 0$ is a constant. Look for separable solutions of the PDE and BCs of the form $u(x,y) = F(x)G(y)$, and find the values of λ for which this BVP has nonzero solutions.

Solution

- Looking for separable solutions, and dividing by u, we get that

  $$-\frac{F''}{F} - \frac{G''}{G} = \lambda.$$

  Introducing a separation constant μ, we then have

  $$F'' + \mu F = 0, \qquad G'' + (\lambda - \mu)G = 0.$$

- The homogeneous boundary conditions are satisfied if

  $$F(0) = 0, \quad F(a) = 0, \qquad G(0) = 0, \quad G(b) = 0.$$

- To get nonzero solutions, we must have (up to constant factors)

  $$\begin{aligned} F(x) &= \sin\left(\frac{m\pi x}{a}\right), \qquad & \mu &= \frac{m^2\pi^2}{a^2}, \\ G(y) &= \sin\left(\frac{n\pi y}{b}\right), \qquad & \lambda - \mu &= \frac{n^2\pi^2}{b^2}, \end{aligned}$$

  where $m, n \in \mathbb{N}$.

- It follows that we get nonzero solutions, labeled by $m, n \in \mathbb{N}$, if

  $$u(x,y) = C \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \qquad \lambda = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

Remark. The values of λ for which we have nonzero solutions are the eigenvalues of the (negative) Laplacian with Dirichlet BCs on the rectangle.

3. [20%] Consider the following integro-differential IVP for $u(x,t)$:

$$u_t(x,t) + \int_{\infty}^{\infty} e^{-|x-y|} u(y,t)\, dy = 0 \qquad -\infty < x < \infty, \quad t > 0,$$

$$u(x,0) = f(x), \qquad -\infty < x < \infty.$$

Solve this IVP by taking Fourier transforms in x, and show that the solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy,$$

where (using $\mathcal{F}^{-1}$ to denote the inverse Fourier transform with respect to x)

$$G(x,t) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left[e^{-2t/(1+\xi^2)}\right].$$

Solution

- Let $\hat{u}(\xi,t) = \mathcal{F}[u(x,t)]$ be the Fourier transform of u with respect to x. Taking the Fourier transform of the evolution equation for u, and using the convolution theorem together with the Fourier transform

$$\mathcal{F}[e^{-|x|}] = \frac{2}{\sqrt{2\pi}} \left(\frac{1}{1+\xi^2}\right),$$

  we get that

$$\hat{u}_t(\xi,t) + \sqrt{2\pi} \cdot \frac{2}{\sqrt{2\pi}} \left(\frac{1}{1+\xi^2}\right) \hat{u}(\xi,t) = 0,$$

  with the initial condition $\hat{u}(\xi,0) = \hat{f}(\xi)$.

- The solution of this ODE in t is

$$\hat{u}(\xi,t) = \hat{f}(\xi) e^{-2t/(1+\xi^2)}.$$

- Using the convolution theorem to invert the Fourier transform, we get that

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t) f(y)\, dy, \quad G(x,t) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left[e^{-2t/(1+\xi^2)}\right].$$

Remark. The Fourier transform can often be used to simplify equations that involve a convolution.

4. [15%] (a) Suppose that $f(x)$ satisfies Airy's equation

$$f''(x) - xf(x) = 0. \tag{1}$$

Take the Fourier transform of this equation and find an ODE for $\hat{f}(\xi)$.
(b) Solve the ODE for $\hat{f}(\xi)$.
(c) How many constants of integration does your solution in (b) have? How many constants of integration does the general solution of (1) have. Can you explain the discrepancy?

Solution

- (a) Let $\hat{f}(\xi) = \mathcal{F}[f(x)]$. Taking the Fourier transform of Airy's equation and using the properties $d/dx \to i\xi$ and $x \to id/d\xi$, we get that

$$i\hat{f}'(\xi) + \xi^2 \hat{f}(\xi) = 0.$$

- (b) The solution of this first-order ODE is

$$\hat{f}(\xi) = ce^{i\xi^3/3},$$

where c is a constant of integration.

- (c) This solution for the Fourier transform has one constant of integration. Equation (1) is a linear, homogeneous, second order ODE. The general solution has the form

$$f(x) = c_1 \operatorname{Ai}(x) + c_2 \operatorname{Bi}(x),$$

where $\{\operatorname{Ai}, \operatorname{Bi}\}$ is a fundamental pair of linearly independent solutions, which involves two constants of integration.

- The explanation of the discrepancy is that some solutions of the Airy equation grow exponentially as $x \to \infty$, and this is too fast for their Fourier transforms to be well-defined (even as tempered distributions). As a result, these solutions are not obtained by the use of Fourier transforms.

Remark. Solutions of Airy's equation that decay exponentially as $x \to \infty$ are proportional to the Airy function $\operatorname{Ai}(x)$, which is defined by

$$\operatorname{Ai}(x) = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}[e^{i\xi^3/3}], \qquad \operatorname{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi^3/3 + x\xi)} \, d\xi.$$

5. [20%] (a) Look for similarity solutions of following PDE of the given form

$$u_t = (uu_x)_x, \qquad u(x,t) = \frac{1}{t^{1/3}} F\left(\frac{x}{t^{1/3}}\right).$$

Derive an ODE for $F(\xi)$, and show that a solution of the ODE is given by

$$F(\xi) = \begin{cases} \frac{1}{6}\left(a^2 - \xi^2\right) & \text{if } |\xi| < a \\ 0 & \text{if } |\xi| \geq a \end{cases}$$

where $a > 0$ is a constant of integration.
(b) Find the value of a for which the similarity solution corresponding to this solution for F satisfies $u(x,t) \to \delta(x)$ as $t \to 0^+$.

Solution

- We have

$$\begin{aligned} u_t(x,t) &= -\frac{1}{3} t^{-4/3} \left(\xi F' + F\right) \\ &= -\frac{1}{3} t^{-4/3} \left(\xi F\right)', \\ (uu_x)_x &= t^{-4/3} \left(F F'\right)'. \end{aligned}$$

  Thus, the similarity solution satisfies the PDE if

$$(FF')' + \frac{1}{3}(\xi F)' = 0.$$

- Integrating this ODE once, and setting the constant of integration to 0, we get that

$$F\left(F' + \frac{1}{3}\xi\right) = 0.$$

  This equation is satisfied either if $F = 0$ or

$$F' + \frac{1}{3}\xi = 0,$$

  with solutions

$$F(\xi) = \frac{1}{6}\left(a^2 - \xi^2\right).$$

- Joining this solution continuously to the solution $F = 0$ at $\xi = \pm a$, we get the solution given in the question. (In fact, one can show that this gives a weak solution of the PDE, even though $F(\xi)$ is not pointwise differentiable at $\xi = \pm a$.)

- (b) The similarity solution for $u(x,t)$ is zero for $|x| \geq at^{1/3}$, so it tends uniformly to 0 as $t \to 0^+$ away from the origin. In order for it to approach a δ-function as $t \to 0^+$, we need

$$\int_{-\infty}^{\infty} u(x,t)\, dx \to 1 \qquad \text{as } t \to 0^+.$$

- Making the change of variables $x = t^{1/3}\xi$, we compute that

$$\begin{aligned}
\int_{-\infty}^{\infty} u(x,t)\, dx &= \frac{1}{t^{1/3}} \int_{-\infty}^{\infty} F\left(\frac{x}{t^{1/3}}\right)\, dx \\
&= \int_{-\infty}^{\infty} F(\xi)\, d\xi \\
&= 2 \cdot \frac{1}{6} \int_0^a \left(a^2 - \xi^2\right)\, d\xi \\
&= \frac{2a^3}{9},
\end{aligned}$$

so $u(x,t) \to \delta(x)$ as $t \to 0^+$ if $a = (9/2)^{1/3}$.

Remark. The PDE is this problem is a nonlinear diffusion equation called the porous medium equation. The diffusivity u vanishes at $u = 0$, which models a situation in which water does not diffuse through dry soil. Nevertheless, as illustrated by the similarity solution, one gets wetting fronts that spread out through the porous medium at finite speed — a surprising phenomenon for a diffusion equation.

This similarity solution (or the cylindrically symmetric version of it) was obtained by Zeldovitch and Kompaneets (1950) and Barenblatt (1952).

6 [15%] Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function on $\mathbb{R}^2$, and denote by $\bar{D}_R(x) = \{y \in \mathbb{R}^2 : |x - y| \leq R\}$ the closed disc of radius R, center $x \in \mathbb{R}^2$.
(a) Explain why the following identity is true for any $x \in \mathbb{R}^2$ and $R > 0$ (you don't need to give a detailed derivation):

$$u(x) - u(0) = \frac{1}{\pi R^2}\left(\int_{\bar{D}_R(x)} u(y)\, dy - \int_{\bar{D}_R(0)} u(y)\, dy\right).$$

(b) Recall that $u : \mathbb{R}^2 \to \mathbb{R}$ is bounded if there exists a constant $M \geq 0$ such that $|u(x)| \leq M$ for all $x \in \mathbb{R}^2$. Deduce from (a) that a bounded harmonic function on $\mathbb{R}^2$ is constant.

Solution

- (a) The identity follows from the mean value property of harmonic functions (where we average over discs rather than circles).
- (b) If $|u| \leq M$, It follows from (a) that

$$\begin{aligned} |u(x) - u(0)| &\leq \frac{1}{\pi R^2} \int_{C_R(x)} |u(y)|\, dy \\ &\leq \frac{M}{\pi R^2} a(|x|, R), \end{aligned}$$

  where $C_R(x) = \bar{D}_R(x) \Delta \bar{D}_R(0)$ is the symmetric set difference of the discs of radius R centered at x and 0, and $a(|x|, R)$ is its area.
- It is clear geometrically that $a(|x|, R) = O(R)$ as $R \to \infty$ with $|x|$ fixed. Then, taking the limit of the inequality as $R \to \infty$ with x fixed, we conclude that $|u(x) - u(0)| = 0$ for every $x \in \mathbb{R}^2$, so u is constant.
- In more detail, we have

$$a(|x|, R) = 2\left[\pi R^2 - A(|x|, R)\right]$$

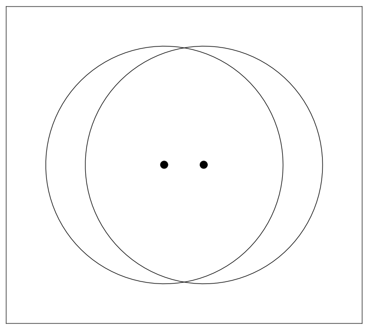
  where $A(d, R)$ is the area enclosed between two circles of radius R with centers a distance d apart. Using the formula

$$A(d, R) = 2R^2 \cos^{-1}\left(\frac{d}{2R}\right) - \frac{1}{2} d\sqrt{4R^2 - d^2},$$

we get that

$$\frac{a(|x|, R)}{R} = 2\pi R - 4R\cos^{-1}\left(\frac{|x|}{2R}\right) + |x|\sqrt{4 - \frac{|x|^2}{R^2}}$$
$$\to 2|x| \qquad \text{as } R \to \infty,$$

and the result follows.



Remark. This short proof of Liouville's theorem for harmonic functions, which we proved in class from Harnack's inequality, is due to Edward Nelson (*Proc. Amer. Math. Soc.*, **12**, 1961). His paper consists of one paragraph (with no equations!) and is as follows:

> Consider a bounded harmonic function on Euclidean space. Since it is harmonic, its value at any point is its average over any sphere, and hence over any ball, with the point as center. Given two points, choose two balls with the given points as centers and of equal radius. If the radius is large enough, the two balls will coincide except for an arbitrarily small proportion of their volume. Since the function is bounded, the averages of it over the two balls are arbitrarily close, and so the function assumes the same value at any two points. Thus a bounded harmonic function on Euclidean space is a constant.