Partial Differential Equations
Math 118B, Winter 2018
Midterm: Solutions

1. [10%] Suppose that $u(r, \theta)$ is harmonic in the open disc of radius 2 and continuous on the closed disc of radius 2, where (r, θ) are polar coordinates. If

$$u(2, \theta) = \pi^2 - \theta^2 \qquad |\theta| \leq \pi,$$

find the value of u at the origin $r = 0$.

Solution

- By the mean value theorem for harmonic functions

$$\begin{aligned} u(0, \theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\pi^2 - \theta^2\right) \, d\theta \\ &= \frac{2}{3}\pi^2. \end{aligned}$$

2. [25%] (a) Let $f : [0,1] \to \mathbb{R}$ is a continuous function. Give a physical interpretation in terms of heat flow of the following boundary value problem for $u(x,y)$ on the unit square:

$$\begin{aligned} &u_{xx} + u_{yy} = 0, && && 0 < x < 1, \quad 0 < y < 1, \\ &u(0,y) = 0, && u(1,y) = 0, && 0 \leq y \leq 1, \\ &u_y(x,0) = 0, && u_y(x,1) = f(x), && 0 \leq x \leq 1. \end{aligned}$$

(b) Use separation of variables to find a formal solution of the boundary value problem.

Solution

- (a) This BVP describes the steady temperature distribution in a square plate of side one. The left and right hand sides of the plate are held at zero temperature; the bottom side is insulated; and the heat flux through the top side is proportional to $-f(x)$.

- (b) The separated solutions of Laplace's equation that satisfy the homogeneous boundary conditions at $x = 0, 1$ and $y = 0$ are proportional to
$$u(x,y) = \sin(n\pi x)\cosh(n\pi y), \qquad n = 1, 2, 3, \ldots$$
Superposing these solutions, we get the formal solution
$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\cosh(n\pi y).$$

- Imposition of the nonhomogeneous boundary condition at $y = 1$ gives
$$f(x) = \sum_{n=1}^{\infty} n\pi B_n \sinh(n\pi)\sin(n\pi x),$$
so, by the formula for Fourier sine series coefficients,
$$n\pi B_n \sinh(n\pi) = b_n, \qquad b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx,$$
and
$$u(x,y) = \sum_{n=1}^{\infty} \frac{b_n}{n\pi\sinh(n\pi)} \sin(n\pi x)\cosh(n\pi y).$$

3. [25%] (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Give a physical interpretation in terms of heat flow of the following initial value problem for $u(x,t)$:

$$
\begin{aligned}
&u_t = u_{xx} + u, && -\infty < x < \infty,\ t > 0, \\
&u(x,0) = f(x), && -\infty < x < \infty.
\end{aligned}
$$

(b) Use Fourier transforms in x to solve the initial value problem for $u(x,t)$. Express your answer as a convolution.

Solution

- (a) This IVP describes heat flow in an infinite rod, with a heat source whose density is proportional to temperature u and inital temperature $f(x)$.

- (b) Let $\hat{u}(\xi,t) = \mathcal{F}[u(x,t)]$ be the Fourier transform of y with respect to x. Then

  $$
  \begin{aligned}
  \hat{u}_t &= \left(1 - \xi^2\right)\hat{u}, \\
  \hat{u}(\xi,0) &= \hat{f}(\xi),
  \end{aligned}
  $$

  whose solution is

  $$\hat{u}(\xi,t) = \hat{f}(\xi)e^{(1-\xi^2)t}.$$

- Since t is a parameter in the Fourier transform with respect to x, we have

  $$
  \begin{aligned}
  \mathcal{F}^{-1}\left[e^{(1-\xi^2)t}\right] &= e^t\mathcal{F}^{-1}\left[e^{-\xi^2 t}\right] \\
  &= e^t \cdot \frac{1}{\sqrt{2t}}e^{-x^2/4t}
  \end{aligned}
  $$

- The convolution theorem then gives that

  $$u(x,t) = \frac{e^t}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-(x-y)^2/4t} f(y)\,dy.$$

- Alternatively, note that writing $u(x,t) = e^t v(x,t)$ in the PDE for u gives $v_t = v_{xx}$, so the solution of the IVP is simply the usual solution of the heat equation multiplied by e^t.

4. [20%] (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a Schwartz function. Use Fourier transforms in x to solve the following initial value problem for $\hat{u}(\xi, t) = \mathcal{F}[u(x,t)]$:

$$\begin{aligned} &u_t = u_{xxx}, && -\infty < x < \infty,\ t > 0, \\ &u(x,0) = f(x), && -\infty < x < \infty. \end{aligned}$$

You do not need to invert the Fourier transform.

(b) Show that for all $-\infty < t < \infty$ the solution satisfies

$$\int_{-\infty}^{\infty} u^2(x,t)\,dx = \int_{-\infty}^{\infty} f^2(x)\,dx.$$

Solution

- Taking Fourier transforms in x, we get that

  $$\begin{aligned} \hat{u}_t &= -i\xi^3 \hat{u}, \\ \hat{u}(\xi, 0) &= \hat{f}(\xi), \end{aligned}$$

  whose solution is

  $$\hat{u}(\xi, t) = \hat{f}(\xi) e^{-i\xi^2 t}.$$

- Note that $e^{-i\xi^2 t}$ is a smooth function of ξ with $|e^{-i\xi^2 t}| = 1$, so $\hat{u}(\cdot, t)$ is a Schwartz function when $\hat{f}(\cdot)$ is a Schwartz function. Since the inverse Fourier transform of a Schwartz function is a Schwartz function, it follows that $u(x,t)$ is a Schwartz function of x for every $t \in \mathbb{R}$.

- Using Parseval's theorem, as the fact that $|e^{-i\xi^2 t}| = 1$, we get

  $$\begin{aligned} \int_{-\infty}^{\infty} u^2(x,t)\,dx &= \int_{-\infty}^{\infty} |\hat{u}|^2(\xi, t)\,d\xi \\ &= \int_{-\infty}^{\infty} |\hat{f}|^2(\xi)\,d\xi \\ &= \int_{-\infty}^{\infty} f^2(x)\,dx. \end{aligned}$$

5. [20%] Let Ω be a bounded open set in $\mathbb{R}^2$ with boundary $\partial\Omega$.

(a) Suppose that $u(x, y)$ is a solution of the PDE

$$u_{xx} + u_{yy} - u = 0.$$

Show that u cannot attain a maximum value at any point of Ω where $u > 0$, or a minimum value at any point of Ω where $u < 0$.

(b) Let $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ be given functions. Show that there is at most one solution of the following boundary value problem:

$$u_{xx} + u_{yy} - u = f \qquad \text{in } \Omega,$$
$$u = g \qquad \text{on } \partial\Omega,$$

Solution

- (a) Suppose that u attain a (local) maximum at some point in Ω. Since Ω is open, the maximum is attained at an interior point, and the second derivative test implies that $u_{xx} \leq 0$ and $u_{yy} \leq 0$ at this point. It follows from the PDE that $u = u_{xx} + u_{yy} \leq 0$, so u cannot attain a maximum at any point where $u > 0$. Similarly, at a minimum we have $u_{xx} \geq 0$ and $u_{yy} \geq 0$, so $u = u_{xx} + u_{yy} \geq 0$, and u cannot attain a minimum at any point in Ω where $u < 0$.

- (b) Suppose that u_1, u_2 are solutions of the BVP. Let $v = u_1 - u_2$. Then, by linearity,

  $$v_{xx} + v_{yy} - v = 0 \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega.$$

  Since $\bar{\Omega} = \Omega \cup \partial\Omega$ is closed and bounded, and a solution v is assumed to be continuous on $\bar{\Omega}$, v attains its maximum value $M = \max_{\bar{\Omega}} v$ at some point in $\bar{\Omega}$. If $M > 0$, then (since $v = 0$ on $\partial\Omega$) the maximum would have to be attained at an interior point in Ω where $v = M > 0$, contradicting (a). Similarly, if $m = \min_{\bar{\Omega}} v < 0$, then the mimimum would have to be attained at an interior point in Ω where $v = m < 0$, also contradicting (a). It follows that the maximum and minimum are attained on the boundary $\partial\Omega$, so $m = M = 0$, which implies that $v = 0$, and $u_1 = u_2$.

Remark. This argument doesn't work for the PDE $u_{xx} + u_{yy} + u = 0$, with the opposite sign on u. In that case, the Dirichlet problem might have non-zero solutions. This corresponds to the fact that the eigenvalues of the Dirichlet problem $-\Delta u = \lambda u$ for the Laplacian are always positive ($\lambda > 0$).