Partial Differential Equations
Math 118B, Winter 2018
Problem Set 6

1. (a) Define

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta}, \qquad \operatorname{sech}\theta = \frac{1}{\cosh\theta}$$

Show that

$$\operatorname{sech}^2\theta = 1 - \tanh^2\theta, \qquad (\tanh\theta)' = \operatorname{sech}^2\theta, \quad (\operatorname{sech}\theta)' = -\tanh\theta\operatorname{sech}\theta,$$

and compare with the corresponding identities for trigonometric functions.
(b) Look for solutions of the KdV traveling wave equation

$$u'' + \frac{1}{2}u^2 - cu = 0,$$

with wave velocity c, of the form

$$u(\theta) = a\operatorname{sech}^2(b\theta).$$

Show that there is a one-parameter family of solutions and determine b, c in terms of a.

2. (a) Suppose that $u(x,t)$ is a smooth solution of the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

that is a Schwartz function of $x \in \mathbb{R}$ for every $t \in \mathbb{R}$. Show that

$$\partial_t\left(u_x^2 - \frac{1}{3}u^3\right) + \partial_x\left(2u_xu_{xxx} - u_{xx}^2 + 2uu_x^2 - u^2u_{xx} - \frac{1}{4}u^4\right) = 0.$$

Hint. To get an equation for $\partial_t(u_x^2)$, differentiation the KdV equation with respect to x and multiply the result by u_x; to get an equation for $\partial_t(u^3)$, multiply the KdV equation by u^2. Combine these equations and express the terms involving spatial derivatives of u as an exact x-derivative.
(b) Deduce that the following integral is conserved on solutions of the KdV equation:

$$\int_{-\infty}^{\infty}\left(u_x^2 - \frac{1}{3}u^3\right)dx = \text{constant}.$$

3. Consider similarity solutions of the linearized KdV equation

$$u_t + u_{xxx} = 0, \qquad u(x,t) = t^{\alpha} F(xt^{\beta}),$$

where α, β are constants and $F : \mathbb{R} \to \mathbb{R}$ is a function of a single variable.

(a) Show that the similarity solutions satisfy the linearized KdV equation if $\beta = -1/3$ and $F(\xi)$ satisfies the ODE

$$F''' - \frac{1}{3}\xi F' + \alpha F = 0.$$

(b) Show that the similarity solutions can satisfy

$$\int_{-\infty}^{\infty} u(x,t)\, dx \to 1 \qquad \text{as } t \to 0$$

only if $\alpha = \beta$.

(c) If $\alpha = \beta = -1/3$ and $F(\xi), F''(\xi) \to 0$ sufficiently rapidly as $\xi \to \infty$, show that

$$F(\xi) = G(3^{-1/3}\xi),$$

where $G(z)$ is a solution of Airy's equation

$$G'' - zG = 0.$$

(d) Deduce that the fundamental solution $g(x,t)$ of the linearized KdV equation, which satisfies

$$\begin{aligned} &g_t + g_{xxx} = 0, \\ &g(x,0) = \delta(x), \qquad g(x,t) \to 0 \text{ as } |x| \to \infty \end{aligned}$$

is given by

$$g(x,t) = \frac{1}{\sqrt[3]{3t}} \mathrm{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right)$$

where $\mathrm{Ai}\,(z)$ is the solution of Airy's equation such that

$$\mathrm{Ai}(z) \to 0 \quad \text{as } |z| \to \infty, \qquad \int_{-\infty}^{\infty} \mathrm{Ai}(z)\, dz = 1.$$

4. Burgers equation is

$$u_t + uu_x = \nu u_{xx}$$

where ν is a constant (with the physical interpretation of a viscosity). Look for traveling wave solutions of Burgers equation

$$u(x,t) = U(x - ct)$$

such that

$$U(\xi) \to U_L \quad \text{as } \xi \to -\infty, \qquad U(\xi) \to U_R \quad \text{as } \xi \to +\infty,$$

where U_L, U_R are constants. Show that such a traveling wave exists only if $U_L \geq U_R$, solve explicitly for $U(\xi)$, and express the velcocity c of the traveling wave in terms of U_L, U_R.