Problem Set 1: Solutions

Math 118B: Winter Quarter, 2018

6.1.1

- If
$$\bar{x} = ax + by + e, \qquad \bar{y} = cx + dy + f,$$
then
$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial \bar{x}}{\partial x}\frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial x}\frac{\partial}{\partial \bar{y}} = a\frac{\partial}{\partial \bar{x}} + c\frac{\partial}{\partial \bar{y}}\\ \frac{\partial}{\partial y} &= \frac{\partial \bar{x}}{\partial y}\frac{\partial}{\partial \bar{x}} + \frac{\partial \bar{y}}{\partial y}\frac{\partial}{\partial \bar{y}}, = b\frac{\partial}{\partial \bar{x}} + d\frac{\partial}{\partial \bar{y}},\end{aligned}$$
and
$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \left(a\frac{\partial}{\partial \bar{x}} + c\frac{\partial}{\partial \bar{y}}\right)^2 = a^2\frac{\partial^2}{\partial \bar{x}^2} + 2ac\frac{\partial^2}{\partial \bar{x}\partial \bar{y}} + c^2\frac{\partial^2}{\partial \bar{y}^2},\\ \frac{\partial^2}{\partial y^2} &= \left(b\frac{\partial}{\partial \bar{x}} + d\frac{\partial}{\partial \bar{y}}\right)^2 = b^2\frac{\partial^2}{\partial \bar{x}^2} + 2bd\frac{\partial^2}{\partial \bar{x}\partial \bar{y}} + d^2\frac{\partial^2}{\partial \bar{y}^2}.\end{aligned}$$

- It follows that
$$u_{xx} + u_{yy} = \left(a^2 + b^2\right) u_{\bar{x}\bar{x}} + 2\left(ac + bd\right) u_{\bar{x}\bar{y}} + \left(c^2 + d^2\right) u_{\bar{y}\bar{y}}.$$

6.1.2

- (a) We have

$$\begin{aligned}\Delta\left(c_1u_1+c_2u_2\right) &= \left(c_1u_1+c_2u_2\right)_{xx}+\left(c_1u_1+c_2u_2\right)_{yy}\\ &= c_1u_{1xx}+c_2u_{2xx}+c_1u_{1yy}+c_2u_{2yy}\\ &= c_1\left(u_{1xx}+u_{1yy}\right)+c_2\left(u_{2xx}+u_{2yy}\right)\\ &= c_1\Delta u_1+c_2\Delta u_2,\end{aligned}$$

  meaning that the Laplacian is a linear operator. Hence, if $\Delta u_1 = 0$ and $\Delta u_2 = 0$, then $\Delta(c_1u_1 + c_2u_2) = 0$.

- (b) If $v(x, y) = xu(x, y)$ where u is harmonic, then

$$v_{xx} + v_{yy} = xu_{xx} + 2u_x + xu_{yy} = 2u_x.$$

  Hence, $u_x = 0$ if v is harmonic, which implies that $u = u(y)$. Then $\Delta u = u_{yy} = 0$, so $u(y) = ay + b$.

- (c) For example, $u(x, y) = xy$ is harmonic, but

$$\Delta u^2 = 2(x^2 + y^2) \neq 0,$$

  so u^2 is not harmonic.

6.1.4

- We have

$$(x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$$
$$= u(x,y) + iv(x,y)$$

where

$$u(x,y) = x^3 - 3xy^2, \qquad v(x,y) = 3x^2y - y^3.$$

Then

$$\Delta u = 6x - 6x = 0, \qquad \Delta v = 6y - 6y = 0.$$

6.1.7

- The function $-q$ represents the heat-source density; that is, the rate per unit area at which internal heat sources generate thermal energy.
- The boundary data $-g$ represents the outward heat flux on the boundary; that is, the rate per unit length at which thermal energy leaves the region.
- There can only be a steady state if the rate at which thermal energy is generated inside the region is equal to the rate at which thermal energy leaves the region, meaning that

$$\int_D q\,dxdy = \int_C gds.$$

- This result can be proved directly from the equations. If there is a solution of the Neumann problem, then the divergence theorem gives

$$\begin{aligned}\int_D q\,dxdy &= \int_D \Delta u\,dxdy \\ &= \int_C \nabla u \cdot n\,ds \\ &= \int_C g\,ds.\end{aligned}$$

6.2.5

- Up to a constant factor, the separated solutions of Laplace's equation that vanish at $x = 0, \pi$ and $y = 0$ are given by

$$u(x, y) = \sin(nx)\sinh(ny) \qquad n \in \mathbb{N}.$$

- Superposing these solutions, we get that

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin(nx)\sinh(ny),$$

where the c_n are constants, satisfies Laplace's equation in the unit square and the homogeneous boundary conditions.

- The boundary condition at $y = \pi$ is satisfied if

$$x(x - \pi) = \sum_{n=1}^{\infty} c_n \sin(nx)\sinh(n\pi).$$

Using the expression for Fourier sine coefficients, we have

$$c_n \sinh(n\pi) = \frac{2}{\pi}\int_0^{\pi} x(x - \pi)\sin(nx)\, dx.$$

Integration by parts, or symbolic integration, gives

$$\begin{aligned}\int x(x - \pi)\sin(nx)\, dx &= \frac{1}{n^2}\left[\pi\sin(nx) - 2x\sin(nx)\right] - \frac{2}{n^3}\cos(nx) \\ &\quad + \frac{1}{n}\left[x^2\cos(nx) - \pi x\cos(nx)\right] + C.\end{aligned}$$

Since $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, it follows that

$$\int_0^{\pi} x(x - \pi)\sin(nx)\, dx = \frac{2}{n^3}\left[1 - (-1)^n\right] = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 4/n^3 & \text{if } n \text{ is odd.}\end{cases}$$

- Using the resulting expression for c_n in the series for u, we get that

$$\begin{aligned}u(x, y) &= \frac{8}{\pi}\sum_{n \text{ odd}} \frac{\sin(nx)\sinh(ny)}{n^3\sinh(n\pi)} \\ &= \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{\sin[(2k + 1)x]\sinh[(2k + 1)y]}{(2k + 1)^3\sinh[(2k + 1)\pi]}.\end{aligned}$$

- Let $N \in \mathbb{N}$. For $0 \leq y \leq \pi$, we estimate that

$$\left| u(x,y) - \frac{8}{\pi} \sum_{k=0}^{N} \frac{\sin[(2k+1)x] \sinh[(2k+1)y]}{(2k+1)^3 \sinh[(2k+1)\pi]} \right| \leq \frac{8}{\pi} \sum_{k=N+1}^{\infty} \frac{1}{(2k+1)^3}.$$

- By an integral comparison test,

$$\sum_{k=N+1}^{\infty} \frac{1}{(2k+1)^3} \leq \int_N^{\infty} \frac{1}{(2x+1)^3}\, dx = \frac{1}{4(2N+1)^2},$$

so

$$\left| u(x,y) - \frac{8}{\pi} \sum_{k=0}^{N} \frac{\sin[(2k+1)x] \sinh[(2k+1)y]}{(2k+1)^3 \sinh[(2k+1)\pi]} \right| \leq \frac{2}{\pi(2N+1)^2}.$$

- It follows that the Nth partial sum approximates $u(x,y)$ within ϵ if

$$2N+1 \geq \sqrt{\frac{2}{\pi\epsilon}}.$$

- If $\epsilon = 0.01$, then $\sqrt{2/\pi\epsilon} \approx 7.9788$, so $N = 4$ is sufficient. In fact, numerical computations show that $N = 3$ is enough, with a maximum error less that 0.008. The corresponding sum is shown in Figure 1.

- (c) The Fourier series converges uniformly to $u(x,y)$ on the unit square, and the truncated series approximates the solution everywhere to within 0.01.

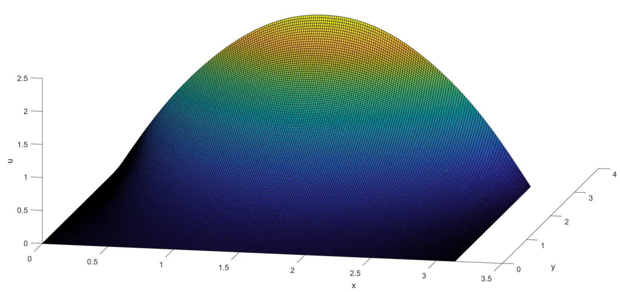

Figure 1: The partial sum in 6.3.5 with $N = 3$.

6.2.8

- (a) The separated solutions of Laplace's equation in Cartesian coordinates whose normal derivatives vanish on $x = 0, L$ and $y = 0$ are proportional to

$$u(x, y) = 1, \qquad u(x, y) = \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right) \quad n \in \mathbb{N}.$$

- Superposing these solutions, we get that

$$u(x, y) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right).$$

- Assuming that the series converges sufficiently rapidly, we can differentiate it term by term, and it follows that

$$u_y(x, y) = \sum_{n=1}^{\infty} c_n \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right).$$

- Imposition of the boundary condition at $y = M$ gives

$$g(x) = \sum_{n=1}^{\infty} c_n \frac{n\pi}{L} \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi M}{L}\right).$$

  The Fourier cosine expansion of g is

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

$$a_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx.$$

- It follows that g must satisfy $a_0 = 0$, meaning that

  $$\int_0^M g(x)\,dx = 0,$$

  and for $n \in \mathbb{N}$ we have

  $$c_n \frac{n\pi}{L} \sinh\left(\frac{n\pi M}{L}\right) = a_n.$$

  The constant c_0 is arbitrary.

- The solutions are then given by

  $$u(x,y) = \frac{1}{2}c_0 + \frac{L}{\pi}\sum_{n=1}^{\infty} \frac{a_n}{n \sinh\left(n\pi M/L\right)} \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right),$$

  where c_0 is an arbitrary constant.

- (b) The condition that $\int_0^M g(x)\,dx = 0$ is the compatibility condition discussed in 6.1.7. Note that a solution does not exist if this condition does not hold; and solutions are not unique if the condition does hold, since we can add an arbitrary constant (which is a solution of the homogeneous Neumann problem) to a solution and get another solution.