Problem Set 2: Solutions
Math 118B: Winter Quarter, 2018

6.3.1

- (e) The general solution obtained by a superposition of separable solutions has the form

$$u(r,\theta) = a_0 + \alpha_0 \log r + \sum_{n=1}^{\infty} u_n(r,\theta) \qquad 1 < r < 2,$$
$$u_n(r,\theta) = \left(a_n r^n + \alpha_n r^{-n}\right)\cos(n\theta) + \left(b_n r^n + \beta_n r^{-n}\right)\sin(n\theta).$$

- Imposition of the boundary condition at $r = 1$ gives

$$a_0 + \sum_{n=1}^{\infty} \left[(a_n + \alpha_n)\cos(n\theta) + (b_n + \beta_n)\sin(n\theta)\right] = 0,$$

  which implies that

$$a_0 = 0, \qquad a_n + \alpha_n = 0, \qquad b_n + \beta_n = 0 \qquad \text{for } n \in \mathbb{N}.$$

- Imposition of the boundary condition at $r = 2$ then gives

$$\alpha_0 \log 2 = 1, \qquad a_1 \cdot 2 + \alpha_1 \cdot 2^{-1} = 3, \qquad b_8 2^8 + \beta_8 2^{-8} = -17,$$
$$a_n 2^n + \alpha_n 2^{-n} = 0, \qquad b_n 2^n + \beta_n 2^{-n} = 0 \qquad \text{otherwise.}$$

- It follows that

$$\alpha_0 = \frac{1}{\log 2}, \qquad a_1 = 2, \qquad \alpha_1 = -2,$$
$$b_8 = -\frac{17}{2^8 + 2^{-8}}, \qquad \beta_8 = \frac{17}{2^8 - 2^{-8}},$$

  and $a_n, \alpha_n, b_n, \beta_n = 0$ otherwise.

- The solution is therefore

$$u(r,\theta) = \frac{\log r}{\log 2} + 2\left(r - r^{-1}\right)\cos\theta - \frac{17\left(r^8 - r^{-8}\right)}{2^8 - 2^{-8}}\sin(8\theta).$$

6.3.3

- (e) The general solution obtained by a superposition of separable solutions that are continuous at $r = 0$ has the form

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right].$$

- Imposition of the boundary condition at $r = 2$ gives

$$1 + 3\cos\theta - 17\sin(8\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[2^n a_n \cos(n\theta) + 2^n b_n \sin(n\theta)\right],$$

  which implies that

$$\frac{1}{2}a_0 = 1, \qquad a_1 = \frac{3}{2}, \qquad b_8 = -\frac{17}{2^8},$$

  and $a_n, b_n, = 0$ otherwise.

- The solution is therefore

$$u(r,\theta) = 1 + \frac{3}{2}r\cos\theta - \frac{17}{2^8}r^8\sin(8\theta).$$

- In this solution, we require that the solution is continuous at $r = 0$. This is not required in Problem 6.3.1(e) because $r = 0$ is not in the domain where we are solving Laplace's equation; instead we impose a boundary condition at $r = 1$.

6.3.8

- The function

  $$u(r, \theta) = P(r, r_3, \theta)$$

  is harmonic inside the disc $0 \le r < r_3$. It follows that for

  $$0 \le r_1 < r_2 < r_3,$$

  the values of u on $r = r_1$ are given in terms of its values on $r = r_2$ by the Poisson integral formula

  $$u(r_1, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P(r_1, r_2, \theta - \phi) u(r_2, \phi) \, d\phi.$$

- In other words,

  $$P(r_1, r_3, \theta) = \frac{1}{2\pi} \int_0^{2\pi} P(r_1, r_2, \theta - \phi) P(r_2, r_3, \phi) \, d\phi.$$

6.3.10

- The general solution of Laplace's equation in the disc, expressed in polar coordinates, has the real form

$$u(r, \theta) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \left[\alpha_n r^n \cos(n\theta) + \beta_n r^n \sin(n\theta)\right],$$

  where the α_n, β_n are constants.

- Assuming that the series converges sufficiently rapidly, we can differentiate it term-by-term to get

$$u_r(r, \theta) = \sum_{n=1}^{\infty} \left[n\alpha_n r^{n-1} \cos(n\theta) + n\beta_n r^{n-1} \sin(n\theta)\right].$$

- The boundary condition $u_r(r_0, \theta) = f(\theta)$, where

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right],$$

  implies that $a_0 = 0$ and

$$\alpha_n = \frac{a_n}{n r_0^{n-1}}, \qquad \beta_n = \frac{b_n}{n r_0^{n-1}}, \qquad n \in \mathbb{N}.$$

  The solution is non-unique up to an arbitrary additive constant $\frac{1}{2}\alpha_0$.

- The condition $a_0 = 0$ is the compatibility condition

$$\int_0^{2\pi} f(\theta)\, d\theta = 0$$

  for the existence of a solution to the Neumann problem (as in Problems 6.1.7, 6.2.8).

6.3.12

- (a) Since $-1 \le \cos\theta \le 1$, we have

$$\begin{aligned} r_0^2 - 2rr_0\cos\theta + r^2 &= (r_0 - r)^2 + 2rr_0(1 - \cos\theta) \ge (r_0 - r)^2, \\ r_0^2 - 2rr_0\cos\theta + r^2 &= (r_0 + r)^2 - 2rr_0(1 + \cos\theta) \le (r_0 + r)^2. \end{aligned}$$

It follows that for $0 \le r < r_0$

$$\begin{aligned} \frac{r_0^2 - r^2}{r_0^2 - 2rr_0\cos\theta + r^2} &\le \frac{r_0^2 - r^2}{(r_0 - r)^2} = \frac{r_0 + r}{r_0 - r}, \\ \frac{r_0^2 - r^2}{r_0^2 - 2rr_0\cos\theta + r^2} &\ge \frac{r_0^2 - r^2}{(r_0 + r)^2} = \frac{r_0 - r}{r_0 + r}. \end{aligned}$$

- (b) If u is harmonic in $r < r_0$ and continuous on $r \le r_0$, then

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r_0^2 - r^2}{r_0^2 - 2rr_0\cos(\theta - \phi) + r^2} u(r_0,\phi)\,d\phi$$

by the Poisson integral formula. If $u \ge 0$, then by using the previous inequalities in this equation, followed by the mean value property of harmonic functions, we get that

$$\begin{aligned} u(r,\theta) &\le \left(\frac{r_0 + r}{r_0 - r}\right)\frac{1}{2\pi}\int_0^{2\pi} u(r_0,\phi)\,d\phi = \left(\frac{r_0 + r}{r_0 - r}\right)u(0), \\ u(r,\theta) &\ge \left(\frac{r_0 - r}{r_0 + r}\right)\frac{1}{2\pi}\int_0^{2\pi} u(r_0,\phi)\,d\phi = \left(\frac{r_0 - r}{r_0 + r}\right)u(0), \end{aligned}$$

which shows that

$$\left(\frac{r_0 - r}{r_0 + r}\right)u(0) \le u(r,\theta) \le \left(\frac{r_0 + r}{r_0 - r}\right)u(0).$$

- (c) Suppose $u \ge 0$ is harmonic on $\mathbb{R}^2$. Fix $r \ge 0$. Then for any $r_0 > r$, we have

$$\left(\frac{1-\rho}{1+\rho}\right)u(0) \le u(r,\theta) \le \left(\frac{1+\rho}{1-\rho}\right)u(0), \qquad 0 < \rho = \frac{r}{r_0} < 1.$$

Taking the limit of this inequality as $\rho \to 0^+$, we get that

$$u(0) \le u(r,\theta) \le u(0),$$

for all (r,θ), meaning that $u = u(0)$ is constant.

- The assumption that $u \geq 0$ is crucial here. For example,

$$u(x, y) = x^2 - y^2$$

  is a nonconstant harmonic function on $\mathbb{R}^2$, but it's not nonnegative.

- (d) Suppose that u is a harmonic function on $\mathbb{R}^2$. Let $M \in \mathbb{R}$ be any constant. If $u \leq M$ on $\mathbb{R}^2$, meaning that u is bounded from above, then $v = M - u$ is a nonnegative harmonic function on $\mathbb{R}^2$, so it is constant. Similarly, if $m \in \mathbb{R}$ and $u \geq m$ on $\mathbb{R}^2$, meaning that u is bounded from below, then $v = u - m$ is a nonnegative harmonic function, so it is constant.

- In particular, if a harmonic function u is bounded on $\mathbb{R}^2$, meaning that $|u(x, y)| \leq M$, then it must be constant.

- If $u : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $u(x_1, y_1) = c_1$, $u(x_2, y_2) = c_2$, then u takes on all values between c_1 and c_2 along a continuous curve joining (x_1, y_1) and (x_2, y_2) by the intermediate value theorem. Hence, if $c \in \mathbb{R}$ and $u(x, y) \neq c$ for any $(x, y) \in \mathbb{R}^2$, then the range of u must be contained in either $(-\infty, c)$ or (c, ∞).

- If $u : \mathbb{R}^2 \to \mathbb{R}$ is a harmonic function that does not take the value c, then the previous results imply that u is either bounded from above or from below, so u is constant. It follows that a nonconstant harmonic function on $\mathbb{R}^2$ takes on all real values.

7.1.1

- (a) The nth complex Fourier coefficient of the function is given for $n \neq 0$ by

$$\begin{aligned} c_n &= \frac{1}{2L}\int_{-L}^{L} xe^{-in\pi x/L}\,dx \\ &= \frac{1}{2L}\left[-\frac{L}{in\pi}xe^{-in\pi x/L} + \frac{L^2}{(in\pi)^2}e^{-in\pi x/L}\right]_{-L}^{L} \\ &= \frac{iL}{2n\pi}\left(e^{-in\pi} + e^{in\pi}\right) \\ &= -\frac{L}{n\pi i}\cos(n\pi) \\ &= \frac{L}{n\pi i}(-1)^{n+1} \end{aligned}$$

and $c_0 = 0$, so

$$x = \frac{L}{\pi i}\sum_{n\neq 0}\frac{(-1)^{n+1}}{n}e^{in\pi x/L} \qquad |x| < L.$$

- (f) We have

$$e^{ax}\cos(bx) = \frac{1}{2}\left[e^{(a+ib)x} + e^{(a-ib)x}\right].$$

For $\lambda \in \mathbb{C}$ consider the Fourier coefficients of the complex-valued function $e^{\lambda x}$:

$$\begin{aligned} \frac{1}{2L}\int_{-L}^{L} e^{\lambda x}e^{-in\pi x/L}\,dx &= \frac{1}{2L}\int_{-L}^{L} e^{(\lambda - in\pi/L)x}\,dx \\ &= \frac{1}{2L}\left[\frac{e^{(\lambda-in\pi/L)x}}{\lambda - in\pi/L}\right]_{-L}^{L} \\ &= \frac{e^{\lambda L - in\pi} - e^{-\lambda L + in\pi}}{2(\lambda L - in\pi)} \end{aligned}$$

Using this result with $\lambda = a + ib$ and $\lambda = a - ib$, we find that the nth Fourier coefficient of $e^{ax}\cos(bx)$ on $[-L, L]$ is given by

$$\begin{aligned} c_n &= \frac{1}{2L}\int_{-L}^{L} e^{ax}\cos(bx)e^{-in\pi x/L}\,dx \\ &= \frac{1}{4}\left[\frac{e^{aL-i(n\pi-bL)} - e^{-aL+i(n\pi-bL)}}{aL - i(n\pi - bL)} + \frac{e^{aL-i(n\pi+bL)} - e^{-aL+i(n\pi+bL)}}{aL - i(n\pi + bL)}\right] \end{aligned}$$

7.1.2

- (a) Collecting terms and using Euler's formula, we have

$$\begin{aligned}\sum_{n=-N}^{n=N} e^{in\theta} &= 1 + \left(e^{i\theta} + e^{-i\theta}\right) + \left(e^{2i\theta} + e^{-2i\theta}\right) + \cdots + \left(e^{iN\theta} + e^{-iN\theta}\right) \\ &= 1 + 2\cos\theta + 2\cos(2\theta) + \cdots + 2\cos(N\theta).\end{aligned}$$

- (b) We have

$$\sum_{n=1}^{N} z^n = z \sum_{n=0}^{N-1} z^n = \frac{z - z^{N+1}}{1 - z}.$$

It follows that

$$\begin{aligned}\sum_{n=-N}^{n=N} e^{in\theta} &= 1 + \sum_{n=1}^{N} \left(e^{i\theta}\right)^n + \sum_{n=1}^{N} \left(e^{-i\theta}\right)^n \\ &= 1 + \frac{e^{i\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} + \frac{e^{-i\theta} - e^{-i(N+1)\theta}}{1 - e^{-i\theta}} \\ &= 1 + \frac{e^{i\theta/2} - e^{i(N+1/2)\theta}}{e^{-i\theta/2} - e^{i\theta/2}} + \frac{e^{-i\theta/2} - e^{-i(N+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}} \\ &= \frac{e^{i(N+1/2)\theta} - e^{-i(N+1/2)\theta}}{e^{i\theta/2} - e^{-i\theta/2}} \\ &= \frac{\sin\left((N+1/2)\theta\right)}{\sin(\theta/2)}.\end{aligned}$$

- (c) From (a) and (b), we get

$$1 + 2\cos\theta + 2\cos(2\theta) + \cdots + 2\cos(N\theta) = \frac{\sin\left((N+1/2)\theta\right)}{\sin(\theta/2)}.$$