Problem Set 3: Solutions

Math 118B: Winter Quarter, 2018

6.4.1

- Let $v = u_2 - u_1$ and $w = u_3 - u_2$. Then v, w are harmonic in D and $v, w \leq 0$ on ∂D. The (weak) maximum principle implies that $v, w \leq 0$ on $\bar{D}$, so $u_1 \leq u_2 \leq u_3$ on $\bar{D}$.

6.4.3

- The functions
$$u(x, y) = A \sin x \sinh y$$
are solutions for any constant A, so solutions are not unique.

- This doesn't contradict Theorem 2 because the domain
$$D = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \infty\right\}$$
is not bounded, as would be required for uniqueness in Theorem 2.

6.4.4

- The functions $u_1(x, y)$, $u_2(x, y)$ are not continuous at $(0, 0)$, so they are not harmonic in
$$D = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\},$$
as would be required for uniqueness in Theorem 2.

6.4.5

- The normal to the graph $z = u(x, y)$ is (u_x, u_y), so the equation of the tangent plane at (x_0, y_0) is $z = \ell(x, y)$ where

$$\ell(x, y) = a(x - x_0) + b(y - y_0) + c,$$
$$a = u_x(x_0, y_0), \quad b = u_y(x_0, y_0), \quad c = u(x_0, y_0).$$

  The function ℓ is linear, so it is harmonic.

- Consider $v = u - \ell$. Then v is harmonic in D and

$$v(x_0, y_0) = 0, \quad v_x(x_0, y_0) = 0, \quad v_y(x_0, y_0) = 0.$$

  Since D is open, there exists $\epsilon > 0$ such that the disc $B_\epsilon(x_0, y_0)$ of radius ϵ and center (x_0, y_0) is contained in D.

- There are two possibilities: (a) v has one sign on $B_\epsilon(x_0, y_0)$, meaning that either $v \geq 0$ or $v \leq 0$; (b) v changes sign on $B_\epsilon(x_0, y_0)$, meaning that there exist $(x_1, y_1), (x_2, y_2) \in B_\epsilon(x_0, y_0)$ such that $v(x_1, y_1) > 0$, $v(x_2, y_2) < 0$.

- In case (a), the harmonic function v has a local min or max at (x_0, y_0), so the strong maximum principle implies that $v = 0$ is constant on $B_\epsilon(x_0, y_0)$, so the graph of u intersects its tangent plane at every point.

- In case (b), there is a continuous curve in $B_\epsilon(x_0, y_0)$ from (x_1, y_1) to (x_2, y_2) that does not pass through (x_0, y_0). The intermediate value theorem implies that v must vanish at some point (x_3, y_3) on this curve, so the tangent plane intersects the graph of u, at least, at two points (x_0, y_0) and (x_3, y_3).

6.4.6

- Let

$$u(x,y) = \log\left(x^2+y^2\right).$$

We compute that, for $(x,y)\neq(0,0)$,

$$u_x = \frac{2x}{x^2+y^2}, \qquad u_{xx} = \frac{-2x^2+2y^2}{x^2+y^2},$$
$$u_y = \frac{2y}{x^2+y^2}, \qquad u_{yy} = \frac{2x^2-2y^2}{x^2+y^2},$$

so $u_{xx}+u_{yy}=0$, and u is harmonic in $\mathbb{R}^2\setminus\{(0,0)\}$.

- If $b<1$, then $u(x,y)$ is harmonic in the disc $B_b(1,0)$ of radius b and center $(1,0)$. The value of u at the center of the disc is

$$u(1,0) = \log 1 = 0.$$

- The boundary of the disc $\partial B_b(1,0)$ has the parametric equation

$$x = 1+b\cos\theta, \qquad y = b\sin\theta \qquad 0\le\theta\le 2\pi.$$

The mean value theorem for harmonic functions implies that

$$u(1,0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(1+b\cos\theta, b\sin\theta\right)\,d\theta,$$

so if $0\le b<1$, it follows that

$$\int_0^{2\pi}\log\left[(1+b\cos\theta)^2+(b\sin\theta)^2\right]\,d\theta = 0.$$

- If $b>2$, then, since $\cos\theta\ge -1$,

$$\begin{aligned}(1+b\cos\theta)^2+(b\sin\theta)^2 &= 1+2b\cos\theta+b^2\\ &\ge 1-2b+b^2\\ &\ge (b-1)^2\\ &> 1,\end{aligned}$$

so $\log\left[(1+b\cos\theta)^2+(b\sin\theta)^2\right]>0$, and

$$\int_0^{2\pi}\log\left[(1+b\cos\theta)^2+(b\sin\theta)^2\right]\,d\theta > 0.$$

6.4.7

- (a) If $u(x,y)$ is a solution of the Dirichlet problem, then $u(x,y)+Ay$ is also a solution for any constant A, so solutions are not unique.
- (b) For $y>0$, we have

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}K(x-s,y)f(s)\,ds, \qquad K(x,y)=\frac{y}{x^2+y^2}>0.$$

- For $(x,y)\neq(0,0)$,

$$\begin{aligned} K_x(x,y)&=\frac{-2xy}{(x^2+y^2)^2}, & K_{xx}(x,y)&=\frac{2y(x^2-y^2)}{(x^2+y^2)^3},\\ K_y(x,y)&=\frac{x^2-y^2}{(x^2+y^2)^2}, & K_{yy}(x,y)&=\frac{2y(y^2-x^2)}{(x^2+y^2)^3}, \end{aligned}$$

so $K_{xx}+K_{yy}=0$. Differentiating under the integral sign, which is justified by the differentiability of the integrand and fact that its derivatives are bounded by an integrable function, we get for $y>0$ that

$$\begin{aligned} u_{xx}+u_{yy}&=\frac{1}{\pi}\int_{-\infty}^{\infty}\left[K_{xx}(x-s,y)+K_{yy}(x-s,y)\right]f(s)\,ds\\ &=0, \end{aligned}$$

so u is harmonic.

- We will show that $u(x,y)\to f(x_0)$ as $(x,y)\to(x_0,0^+)$.
- Suppose that $y>0$ and $\delta>0$. We have

$$\frac{1}{\pi}\int_{x-\delta}^{x+\delta}K(x-s,y)\,ds=\frac{1}{\pi}\int_{-\delta}^{\delta}K(s,y)\,ds,$$

and making a change of variable $s=yt$, we find that

$$\frac{1}{\pi}\int_{-\delta}^{\delta}K(s,y)\,ds=\frac{2}{\pi}\int_{0}^{\delta/y}\frac{1}{1+t^2}\,dt=\frac{2}{\pi}\tan^{-1}\left(\frac{\delta}{y}\right).$$

Hence,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}K(s,y)\,ds=1, \tag{1}$$

$$\lim_{y\to0^+}\frac{1}{\pi}\int_{|s|\geq\delta}K(s,y)\,ds=0. \tag{2}$$

- Using (1), we can write

$$u(x,y) - f(x_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} K(x-s,y)\left[f(s) - f(x_0)\right]\,ds.$$

  We will show that the right-hand side of this equation goes to zero as $(x,y) \to (x_0, 0^+)$.

- Let $\epsilon > 0$ be given. Since f is continuous, there exists $\delta > 0$, depending on x_0 and ϵ, such that

$$|f(s) - f(x_0)| < \frac{\epsilon}{2} \qquad \text{if } |s - x_0| < \delta. \tag{3}$$

  Then

$$\begin{aligned} |u(x,y) - f(x_0)| \le \frac{1}{\pi}\int_{|s-x_0|<\delta} K(x-s,y)\,|f(s) - f(x_0)|\,ds \\ + \frac{1}{\pi}\int_{|s-x_0|\ge\delta} K(x-s,y)\,|f(s) - f(x_0)|\,ds. \end{aligned} \tag{4}$$

- Using (3) and (1), we can estimate the first integral in (4) by

$$\begin{aligned} &\frac{1}{\pi}\int_{|s-x_0|<\delta} K(x-s,y)\,|f(s) - f(x_0)|\,ds \\ &\qquad < \frac{\epsilon}{2}\cdot\frac{1}{\pi}\int_{|s-x_0|<\delta} K(x-s,y)\,ds < \frac{\epsilon}{2}. \end{aligned} \tag{5}$$

- If $|x - x_0| < \delta/4$, then $[x - \delta/4, x + \delta/4] \subset [x_0 - \delta, x_0 + \delta]$, so

$$\{s : |s - x_0| \ge \delta\} \subset \{s : |s - x| \ge \delta/4\},$$

  and

$$\int_{|s-x_0|\ge\delta} K(x-s,y)\,ds \le \int_{|s-x|\ge\delta/4} K(x-s,y)\,ds.$$

- Since $|f(x)| \le M$, we have $|f(s) - f(x_0)| \le 2M$, and can we estimate the second integral in (4) by

$$\frac{1}{\pi}\int_{|s-x_0|\ge\delta} K(x-s,y)\,|f(s) - f(x_0)|\,ds \le 2M\cdot\frac{1}{\pi}\int_{|s|\ge\delta/4} K(s,y)\,ds.$$

  From (2), there exists $\eta > 0$ such that

$$\frac{1}{\pi}\int_{|s|\ge\delta/4} K(s,y)\,ds < \frac{\epsilon}{4M} \qquad \text{if } 0 < y < \eta,$$

in which case

$$\frac{1}{\pi}\int_{|s-x_0|\geq\delta} K(x-s,y)\,|f(s)-f(x_0)|\;ds < \frac{\epsilon}{2}. \tag{6}$$

- Using (5)–(6) in (4), we find that if $|x-x_0| < \delta/4$ and $0 < y < \eta$, then

  $$|u(x,y) - f(x_0)| < \epsilon,$$

  meaning that $u(x,y) \to f(x_0)$ as $(x,y) \to (x_0, 0^+)$.

- This continuity result is stronger than the existence of the "vertical" limit $u(x_0,y) \to f(x_0)$ as $y \to 0^+$ with x_0 fixed, which would be a bit simpler to prove.

6.4.8

- (a) We have

$$\begin{aligned}\frac{1+z}{1-z} &= \frac{(1+z)(1-\bar{z})}{(1-z)(1-\bar{z})} \\ &= \frac{1+z-\bar{z}-|z|^2}{1-z-\bar{z}+|z|^2} \\ &= \frac{1-r^2}{1-2r\cos\theta+r^2} + i\frac{2r\sin\theta}{1-2r\cos\theta+r^2}\end{aligned}$$

  If $w = a + ib$, then $\Im(w^2) = 2ab$, so

$$U(r,\theta) = \Im\left(\frac{1+z}{1-z}\right)^2 = \frac{4r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2}.$$

- After some algebra, one can check that

$$\frac{1}{r}\left(rU_r\right)_r + \frac{1}{r^2}U_{\theta\theta} = 0.$$

  This result also follows from the fact that U is the imaginary part of an analytic function on $\mathbb{C} \setminus \{1\}$.

- (b) If $\theta \not\equiv 0$, then $\lim_{r\to 1^-}\left(1 - 2r\cos\theta + r^2\right) = 2 - 2\cos\theta \neq 0$, so

$$\lim_{r\to 1^-} \frac{4r(1-r^2)\sin\theta}{(1-2r\cos\theta+r^2)^2} = 0.$$

  If $\theta \equiv 0$, then $U(r, 0) = 0$ for all $r < 1$. In either case

$$\lim_{r\to 1^-} U(r,\theta) = 0.$$

- (c) Although all the radial limits of $U(r,\theta)$, in which $r \to 1^-$ with θ fixed, exist and are zero, the function is not continuous at $(1, 0)$. In Cartesian coordinates, we have

$$u(x,y) = \frac{4y(1-x^2-y^2)}{[(x-1)^2+y^2]^2}.$$

  It follows that

$$u(1-y,y) = \frac{2(1-y)}{y^2} \to \infty \qquad \text{as } y \to 0^+.$$

  The uniqueness theorem does not apply because u is not continuous on the closed unit disc $\bar{D}$.

- Note that the existence and equality of the radial limits is not sufficient to imply the continuity of u. This illustrates the necessity in 6.4.7 of proving the limit with $(x, y) \to (x_0, 0)$, not just $(x_0, y) \to (x_0, 0)$ to show the continuity of $u(x, y)$ at $(x_0, 0)$.