Partial Differential Equations
Math 118B, Winter 2018
Problem Set 6: Solutions

1. (a) Define

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta}, \qquad \operatorname{sech}\theta = \frac{1}{\cosh\theta}$$

Show that

$$\operatorname{sech}^2\theta = 1 - \tanh^2\theta, \qquad (\tanh\theta)' = \operatorname{sech}^2\theta, \quad (\operatorname{sech}\theta)' = -\tanh\theta\operatorname{sech}\theta,$$

and compare with the corresponding identities for trigonometric functions.
(b) Look for solutions of the KdV traveling wave equation

$$u'' + \frac{1}{2}u^2 - cu = 0,$$

with wave velocity c, of the form

$$u(\theta) = a\operatorname{sech}^2(b\theta).$$

Show that there is a one-parameter family of solutions and determine b, c in terms of a.

Solution

- (a) Using the definitions

  $$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}, \qquad \sinh\theta = \frac{e^\theta - e^{-\theta}}{2},$$

  one verifies that
  $$\cosh^2\theta - \sinh^2\theta = 1.$$
  Division by $\cosh^2\theta$ then gives

  $$1 - \tanh^2\theta = \operatorname{sech}^2\theta.$$

- It also follows directly from the definitions that

$$(\cosh\theta)' = \sinh\theta, \qquad (\sinh\theta)' = \cosh\theta.$$

  Hence

$$\begin{aligned}
(\tanh\theta)' &= \left(\frac{\sinh\theta}{\cosh\theta}\right)' = \frac{\cosh^2\theta - \sinh^2\theta}{\cosh^2\theta} = \operatorname{sech}^2\theta, \\
(\operatorname{sech}\theta)' &= \left(\frac{1}{\cosh\theta}\right)' = -\frac{\sinh\theta}{\cosh^2\theta} = -\tanh\theta\operatorname{sech}\theta.
\end{aligned}$$

- The sign differences from the corresponding trigonometric identities follow from the fact that $\cos\theta = \cosh(i\theta)$, $\sin\theta = -i\sinh(i\theta)$.

- (b) We have

$$\begin{aligned}
u' &= -2ab\operatorname{sech} b\theta \cdot \tanh b\theta \operatorname{sech} b\theta \\
&= -2ab\tanh b\theta + 2ab\tanh^3 b\theta, \\
u'' &= -2ab^2\operatorname{sech}^2 b\theta + 6ab\operatorname{sech}^2 b\theta\tanh^2 b\theta \\
\\
&= -2ab^2\operatorname{sech}^2 b\theta + 6ab^2\operatorname{sech}^2 b\theta - 6ab^2\operatorname{sech}^4 b\theta,
\end{aligned}$$

  so

$$\begin{aligned}
u'' + \frac{1}{2}u^2 - cu = 4ab^2\operatorname{sech}^2 b\theta - 6ab^2\operatorname{sech}^4 b\theta \\
+ \frac{1}{2}a^2\operatorname{sech}^4 b\theta - ac\operatorname{sech}^2 b\theta.
\end{aligned}$$

- We get a solution if $a = 12b^2$, $c = 4b^2$, meaning that

$$b = \sqrt{\frac{a}{12}}, \qquad c = \frac{1}{3}a.$$

2. (a) Suppose that $u(x,t)$ is a smooth solution of the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

that is a Schwartz function of $x \in \mathbb{R}$ for every $t \in \mathbb{R}$. Show that

$$\partial_t \left(u_x^2 - \frac{1}{3}u^3 \right) + \partial_x \left(2u_x u_{xxx} - u_{xx}^2 + 2uu_x^2 - u^2 u_{xx} - \frac{1}{4}u^4 \right) = 0.$$

(b) Deduce that the following integral is conserved on solutions of the KdV equation:

$$\int_{-\infty}^{\infty} \left(u_x^2 - \frac{1}{3}u^3 \right) dx = \text{constant}.$$

Solution

- (a) Differentiating the equation with respect to x and multipying the result by u_x, we have

  $$u_x u_{xt} + uu_x u_{xx} + u_x^3 + u_x u_{xxxx} = 0.$$

  We can write

  $$\begin{aligned} u_x u_{xt} &= \frac{1}{2}\partial_t \left(u_x^2\right), \\ uu_x u_{xx} &= \frac{1}{2}u\partial_x \left(u_x^2\right) \\ &= \frac{1}{2}\partial_x \left(uu_x^2\right) - \frac{1}{2}u_x^3, \\ u_x u_{xxxx} &= \partial_x \left(u_x u_{xxx}\right) - u_{xx}u_{xxx} \\ &= \partial_x \left(u_x u_{xxx} - \frac{1}{2}u_{xx}^2 \right) \end{aligned}$$

  It follows that

  $$\partial_t \left(u_x^2\right) + \partial_x \left(2u_x u_{xxx} - u_{xx}^2 + uu_x^2\right) + u_x^3 = 0. \tag{1}$$

- Multiplying the equation by u^2, we have

  $$u^2 u_t + u^3 u_x + u^2 u_{xxx} = 0.$$

which we can write as

$$\partial_t \left(\frac{1}{3} u^3 \right) + \partial_x \left(\frac{1}{4} u^4 + u^2 u_{xx} - u u_x^2 \right) + u_x^3 = 0. \tag{2}$$

Subtracting (2) from (1), we get the result.

- (b) Integration of the conservation law over $-\infty < x < \infty$ gives

$$\frac{d}{dt} \int_{-\infty}^{\infty} \left(u_x^2 - \frac{1}{3} u^3 \right) \, dx = 0,$$

since, for Schwartz functions $u(x,t)$ of x,

$$\begin{aligned} &\int_{-\infty}^{\infty} \partial_x \left(2u_x u_{xxx} - u_{xx}^2 + 2uu_x^2 - u^2 u_{xx} - \frac{1}{4} u^4 \right) \, dx \\ &\qquad = \left[2u_x u_{xxx} - u_{xx}^2 + 2uu_x^2 - u^2 u_{xx} - \frac{1}{4} u^4 \right]_{-\infty}^{\infty} \\ &\qquad = 0. \end{aligned}$$

- It follows that

$$\int_{-\infty}^{\infty} \left(u_x^2 - \frac{1}{3} u^3 \right) \, dx = \text{constant}$$

is a constant independent of time.

Remark. It turns out that the KdV equation has infinitely many conserved quantities, which is one indication of its complete integrability.

3. Consider similarity solutions of the linearized KdV equation

$$u_t + u_{xxx} = 0, \qquad u(x,t) = t^\alpha F(xt^\beta),$$

where α, β are constants and $F : \mathbb{R} \to \mathbb{R}$ is a function of a single variable.
(a) Show that the similarity solutions satisfy the linearized KdV equation if $\beta = -1/3$ and $F(\xi)$ satisfies the ODE

$$F''' - \frac{1}{3}\xi F' + \alpha F = 0.$$

(b) Show that the similarity solutions can satisfy $\int_{-\infty}^{\infty} u(x,t)\,dx \to 1$ as $t \to 0$ only if $\alpha = \beta$.
(c) If $\alpha = \beta = -1/3$ and $F(\xi), F''(\xi) \to 0$ sufficiently rapidly as $\xi \to \infty$, show that

$$F(\xi) = G(3^{-1/3}\xi),$$

where $G(z)$ is a solution of Airy's equation

$$G'' - zG = 0.$$

(d) Deduce that the fundamental solution $g(x,t)$ of the linearized KdV equation, which satisfies

$$\begin{aligned} &g_t + g_{xxx} = 0, \\ &g(x,0) = \delta(x), \qquad g(x,t) \to 0 \text{ as } |x| \to \infty \end{aligned}$$

is given by

$$g(x,t) = \frac{1}{\sqrt[3]{3t}} \mathrm{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right)$$

where $\mathrm{Ai}\,(z)$ is the solution of Airy's equation such that

$$\mathrm{Ai}(z) \to 0 \quad \text{as } |z| \to \infty, \qquad \int_{-\infty}^{\infty} \mathrm{Ai}(z)\,dz = 1.$$

Solution

- (a) If $u = t^\alpha F(xt^\beta)$, then

$$\begin{aligned} u_t &= \alpha t^{\alpha-1} F + \beta x t^{alpha+\beta-1} F' \\ &= t^{\alpha-1}\left(\alpha F + \beta \xi F'\right), \\ u_{xxx} &= t^{\alpha+3\beta} F''', \end{aligned}$$

  where $\xi = xt^\beta$.

- Hence, the similarity solution is compatible with the PDE if

$$\alpha - 1 = \alpha + 3\beta,$$

or $\beta = -1/3$, in which case $F(\xi)$ satisfies

$$F''' - \frac{1}{3}\xi F' + \alpha F = 0.$$

- (b) Making the change of variables $\xi = xt^\beta$ in the integral of the similarity solution, we have

$$\int_{-\infty}^{\infty} u(x,t)\,dx = t^\alpha \int_{-\infty}^{\infty} F(xt^\beta)\,dx$$
$$= t^{\alpha-\beta} \int_{-\infty}^{\infty} F(\xi)\,d\xi,$$

and this can only have a finite, non-zero limit as $t \to 0$ if $\alpha = \beta$.

- (c) If $\alpha = \beta = -1/3$, then the ODE for F becomes

$$F''' - \frac{1}{3}(\xi F)' = 0,$$

which can be integrated to get

$$F'' - \frac{1}{3}\xi F = 0.$$

The constant of integration is zero since we assume that $F'', \xi F \to 0$ as $\xi \to \infty$.

- Introducing a rescaled independent variable $z = 3^{-1/3}\xi$, we get that $F(\xi) = G(z)$ where $G'' - zG = 0$, as stated.

- Taking $G(z) = \mathrm{Ai}(z)$, we have that

$$u(x,t) = \frac{1}{\sqrt[3]{3t}}\mathrm{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right)$$

is a solution of the linearized KdV equation $u_t + u_{xxx} = 0$.

- Moreover for all $t \neq 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} u(x,t)\,dx &= \frac{1}{\sqrt[3]{3t}}\mathrm{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right)\,dx \\ &= \mathrm{Ai}\left(\xi\right)\,d\xi \\ &= 1\end{aligned}$$

  This result is compatible with $u(x,t) \rightharpoonup \delta(x)$ as $t \to 0$, but the limit is rather singular, and we need to know more about the Airy function (and distribution theory) than what is given in the question to show this is actually the case.

4. Burgers equation is

$$u_t + uu_x = \nu u_{xx}$$

where $\nu > 0$ is a constant (with the physical interpretation of a viscosity). Look for traveling wave solutions of Burgers equation

$$u(x,t) = U(x - ct)$$

such that

$$U(\xi) \to U_L \quad \text{as } \xi \to -\infty, \qquad U(\xi) \to U_R \quad \text{as } \xi \to +\infty,$$

where U_L, U_R are constants. Show that such a traveling wave exists only if $U_L \geq U_R$, solve explicitly for $U(\xi)$, and express the velcocity c of the traveling wave in terms of U_L, U_R.

Solution

- The traveling wave equation is

  $$\nu U'' = UU' - cU'.$$

  We can integrate this up once to get

  $$\nu U' = \frac{1}{2}U^2 - cU + b,$$

  where b is a constant of integration.

- If $U(\xi)$ approaches a constant U_R as $\xi \to \infty$, then we must have $U'(\xi) \to 0$ as $\xi \to \infty$, so U_R must be a root of $\frac{1}{2}U^2 - cU + b = 0$; similarly, U_L must also be a root. It follows that

  $$\frac{1}{2}U^2 - cU + b = \frac{1}{2}(U - U_L)(U - U_R),$$

  which implies that $b = \frac{1}{2}U_L U_R$ and

  $$c = \frac{1}{2}\left(U_L + U_R\right).$$

  That is, the speed of the traveling wave is the average of its limiting values.

- Looking at the phase line of the ODE
$$\nu U' = \frac{1}{2}(U - U_L)(U - U_R),$$
we see that U must be between U_L and U_R to get a bounded traveling wave. It that case $U' < 0$, where we assume that ν is positive, and we only have $U(\xi) \to U_L$ as $\xi \to -\infty$ and $U(\xi) \to U_R$ as $\xi \to \infty$ if $U_L > U_R$. (If $U_L = U_R$, then we get a constant solution.)

- Solving the ODE by separation of variables, we get
$$\int \frac{dU}{(U - U_L)(U - U_R)} = \frac{1}{2\nu} \int d\xi,$$
which gives (by partial fractions)
$$\frac{1}{U_L - U_R} \log \left| \frac{U - U_R}{U - U_L} \right| = \frac{1}{2\nu}(\xi - \xi_0).$$
Here, ξ_0 is a constant of integration that corresponds to a spatial translation of the traveling wave, and we set $\xi_0 = 0$ without loss of generality.

- For $U_R < U < U_L$, we get that
$$\frac{U - U_R}{U_L - U} = e^{\xi/2a}, \qquad a = \frac{\nu}{\sigma}, \quad \sigma = U_L - U_R,$$
so
$$U(\xi) = \frac{U_R + U_L e^{\xi/2a}}{1 + e^{\xi/2a}}.$$
This solution can also be written as
$$U(\xi) = c + \frac{1}{2}\sigma \tanh\left(\frac{\xi}{4a}\right).$$

Remark. This traveling wave solution describes the profile (called the Taylor profile) of a weak viscous shock. Only compressive shocks for the inviscid Burgers equation with $U_L > U_R$ can arise as a zero viscosity limit; expansion shocks with $U_L < U_R$ are non-physical. Note that the width a of the viscous shock is proportional to the viscosity ν and inversely proportional to the shock strength σ, as could have been predicted from dimensional analysis:
$$[\nu] = \frac{L^2}{T}, \qquad [\sigma] = \frac{L}{T}, \qquad [a] = L.$$