Ordinary Differential Equations

Math 119B, Spring 2017

Midterm: Solutions

1. [15%] Show that the following system has no closed orbits:

$$\dot{x} = \cos y + y \cos x, \qquad \dot{y} = \sin x - x \sin y.$$

Solution.

- Since

$$\frac{\partial}{\partial y}(\cos y + y \cos x) = -\sin y + \cos x$$
$$\frac{\partial}{\partial x}(\sin x - x \sin y) = \cos x - \sin y$$

  are equal (and $\mathbb{R}^2$ is simply connected), the system is a gradient system (with potential $V(x, y) = -x \cos y - y \sin x$), so it does not have any closed orbits.

2. [25%] (a) State the Poincaré-Bendixson theorem.
(b) Show that the system

$$\dot{x} = y, \qquad \dot{y} = -x + y(4 - x^2 - 4y^2)$$

has at least one closed orbit in the annulus $1 \leq x^2 + y^2 \leq 4$.

Solution.

- (a) **Poincaré-Bendixson Theorem**: Suppose that $R \subset \mathbb{R}^2$ is a closed, bounded subset of the plane. If a smooth, planar dynamical system has no fixed points in R and there is a trajectory $\mathbf{x}(t)$ that enters R and remains in R for all subsequent times t, then R contains a closed orbit. Moreover, the trajectory $\mathbf{x}(t)$ either is a closed orbit or it approaches a closed orbit as $t \to \infty$. (More precisely, the ω-limit set of $\mathbf{x}(t)$ is a closed orbit.)

- (b) We compute that if $(x(t), y(t))$ is a solution of the ODE, then

$$\frac{1}{2}\frac{d}{dt}\left(x^2 + y^2\right) = x\dot{x} + y\dot{y} = y^2(4 - x^2 - 4y^2).$$

- On $x^2 + y^2 = 1$, we have $y^2 = 1 - x^2$ and

$$y^2(4 - x^2 - 4y^2) = 3x^2y^2 \geq 0,$$

  so any trajectory with $x^2 + y^2 = 1$ at some initial time must remain in the region $x^2 + y^2 \geq 1$ for all subsequent times.

- On $x^2 + y^2 = 4$, we have $x^2 = 4 - y^2$ and

$$y^2(4 - x^2 - 4y^2) = -3y^4 \leq 0,$$

  so any trajectory with $x^2 + y^2 = 4$ at some initial time must remain in the region $x^2 + y^2 \leq 4$ for all subsequent times.

- It follows that the annulus $1 \leq x^2 + y^2 \leq 4$ is a trapping region for the flow. The only fixed point of the system is $(x, y) = (0, 0)$, so this region doesn't contain any fixed points. The Poincaré-Bendixson theorem then implies that the annulus must contain a closed orbit.

3. [30%] Consider the system

$$\dot{x} = y, \qquad \dot{y} = \mu + 2x + x^2 - xy.$$

where μ is a parameter.

(a) Find the fixed points and classify them.

(b) For what value of μ does a bifurcation occur in this system? What kind of bifurcation is it?

Solution.

- The fixed points $(\bar{x}, \bar{y})$ satisfy $\bar{y} = 0$ and $\bar{x}^2 + 2\bar{x} + \mu = 0$, so

  $$\bar{x} = -1 \pm \sqrt{1-\mu}.$$

  For $\mu < 1$, there are two fixed points

  $$(\bar{x}, \bar{y}) = \left(-1 \pm \sqrt{1-\mu}, 0\right).$$

  For $\mu = 1$, there is one fixed point

  $$(\bar{x}, \bar{y}) = (-1, 0),$$

  and for $\mu > 1$, there are no fixed points.

- The Jacobian matrix J of the system is

  $$J(x, y) = \begin{pmatrix} 0 & 1 \\ 2(1+x) - y & -x \end{pmatrix}$$

- At the fixed point,

  $$J(\bar{x}, 0) = \begin{pmatrix} 0 & 1 \\ 2(1+\bar{x}) & -\bar{x} \end{pmatrix},$$

  and, writing $\tau = \mathrm{tr} J$, $\Delta = \det J$, we have

  $$\tau = -\bar{x}, \qquad \Delta = -2\,(1+\bar{x})$$

- If $\bar{x} = -1 + \sqrt{1-\mu}$, then $\Delta < 0$, so the fixed point is a saddle point.

- If $\bar{x} = -1 - \sqrt{1-\mu}$, then $\Delta > 0$ and $\tau > 0$, so the fixed point is an unstable node or spiral. Moreover, we have

$$\tau^2 - 4\Delta = \bar{x}^2 + 8\bar{x} + 8 = (\bar{x} + 4)^2 - 8.$$

- The fixed point is an unstable spiral if $\tau^2 - 4\Delta < 0$, or $|\bar{x} + 4| < \sqrt{8}$ and

$$-4 - \sqrt{8} < \bar{x} < -4 + \sqrt{8}.$$

  This inequality implies that

$$1 - \left(3 + \sqrt{8}\right)^2 < \mu < 1 - \left(3 - \sqrt{8}\right)^2.$$

- On the other hand, the fixed point is an unstable node if

$$1 - \left(3 - \sqrt{8}\right)^2 < \mu < 1 \quad \text{or} \quad \mu < 1 - \left(3 + \sqrt{8}\right)^2.$$

- (b) A saddle-node bifurcation occurs at $(x, y; \mu) = (-1, 0; 1)$, where the unstable node coalesces with the saddle point.

4. [30%] Consider the system

$$\dot{x} = x\left(x - x^2 - y\right), \qquad \dot{y} = y\,(x - \mu)\,.$$

where μ is a parameter.

(a) Find the fixed point for which both x and y are nonzero. Determine its stability.

(b) For what value of μ does a bifurcation occur at the fixed point in (a)? What kind of bifurcation is it?

Solution.

- (a) The fixed point $(\bar{x}, \bar{y})$ with $\bar{x}, \bar{y} \neq 0$ is

  $$(\bar{x}, \bar{y}) = \left(\mu, \mu - \mu^2\right),$$

  where $\mu \neq 0, 1$.

- The Jacobian is

  $$J(x, y) = \begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - \mu \end{pmatrix},$$

  so

  $$J(\bar{x}, \bar{y}) = \begin{pmatrix} \mu - 2\mu^2 & -\mu \\ \mu - \mu^2 & 0 \end{pmatrix}.$$

  The trace and determinant of this matrix are

  $$\tau = \mu(1 - 2\mu), \qquad \Delta = \mu^2\,(1 - \mu)\,.$$

- The fixed point is unstable for $0 < \mu < 1/2$ (since $\tau > 0$) and $\mu > 1$ (since $\Delta < 0$).

- The fixed point is stable for $\mu < 0$ and $1/2 < \mu < 1$ (since $\Delta > 0$, $\tau < 0$).

- Excluding the values $\mu = 0, 1$ where one of $\bar{x}$ or $\bar{y}$ is zero, and equilibrium bifurcations occur, we see that the fixed point changes stability at $\mu = 1/2$.

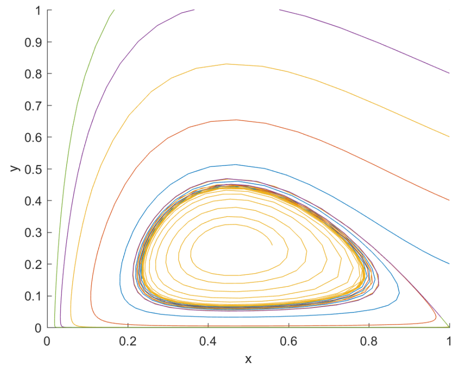

Figure 1: Phase plane for $\mu = 0.46$.

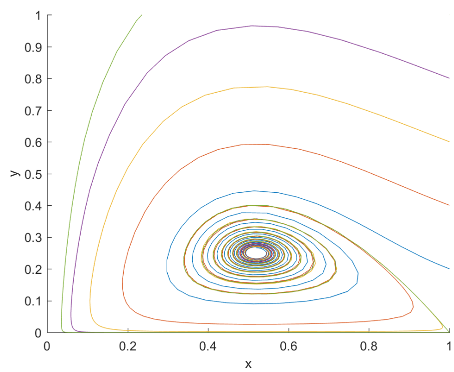

Figure 2: Phase plane for $\mu = 0.52$.

- As μ increases through $1/2$, the trace τ decreases through 0 at a nonzero, positive value of $\Delta = 1/8$. It follows that the fixed point changes from an unstable spiral point to a stable spiral point, and the Jacobian matrix has a complex conjugate pair of eigenvalues that cross from the right-half to the left-half of the complex plane. The Hopf bifurcation theorem implies (subject to a genericity condition on the nonlinearity) that a Hopf bifurcation takes place at $(x, y; \mu) = (1/2, 1/4; 1/2)$. Numerical solutions show that there is a stable limit cycle for μ close to $1/2$ and $\mu < 1/2$.