Sample Problems: Midterm 1

1. (a) Is the following system a gradient system? A Hamiltonian system?

$$\dot{x} = -y + x^3, \qquad \dot{y} = x + y^5.$$

(b) Show that the system has no periodic orbits.

2. Consider the system

$$\dot{x} = x\left[x(1-x) - y\right], \qquad \dot{y} = y(x - \mu),$$

where μ is a parameter.
(a) Find the fixed points and determine their linearized stability.
(b) Find the bifurcation points and classify the bifurcations.

3. (a) Write the scalar ODE

$$\ddot{x} + \mu\left(x^2 - 4\right)\dot{x} + x = 1$$

as a first order system for (x, y) where

$$y = \frac{\dot{x}}{\mu} + \frac{x^3}{3} - 4x.$$

(b) Give a qualitative argument for the existence of a limit cycle solution when μ is large and positive. Sketch the limit cycle in the phase plane.
(c) Extra credit: Estimate the period of the limit cycle for large values of μ.

4. Consider the nonlinear system

$$\dot{x} = y - x\left\{\left(x^2+y^2\right)^4 - \mu\left[\left(x^2+y^2\right)^2 - 1\right] - 1\right\}$$
$$\dot{y} = -x - y\left\{\left(x^2+y^2\right)^4 - \mu\left[\left(x^2+y^2\right)^2 - 1\right] - 1\right\}$$

(a) Write the system in polar coordinates (r, θ).
(b) State the Poincaré-Bendixson theorem.
(c) For $0 \le \mu < 1$, show that $1/2 < r < 2$ is a trapping region, and deduce that it contains a limit cycle.
(d) Show that a Hopf bifurcation occurs at $\mu = 1$. Is it subcritical or supercritical?