Sample Problems: Midterm 1

1. (a) Is the following system a gradient system? A Hamiltonian system?

$$\dot{x} = -y + x^3, \qquad \dot{y} = x + y^5.$$

(b) Show that the system has no periodic orbits.

Solution.

- (a) If the system
$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$
is a gradient system with
$$f = -\frac{\partial V}{\partial x}, \qquad g = -\frac{\partial V}{\partial y},$$
then
$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$
Here $f(x, y) = -y + x^3$, $g(x, y) = x + y^5$ and
$$\frac{\partial f}{\partial y} = -1, \qquad \frac{\partial g}{\partial x} = 1,$$
so the system is not a gradient system.

- If the system is a Hamiltonian system with
$$f = \frac{\partial H}{\partial y}, \qquad g = -\frac{\partial H}{\partial x},$$
then
$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0,$$
but here
$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 3x^2 + 5y^4,$$
so the system is not a Hamiltonian system.

- (b) On any orbit (x, y), except the fixed point $(x, y) = (0, 0)$, we have

$$\begin{aligned}
\frac{d}{dt}\left(x^2 + y^2\right) &= 2x\dot{x} + 2y\dot{y} \\
&= 2x\left(-y + x^3\right) + 2y\left(x + y^5\right) \\
&= 2x^4 + 2y^6 \\
&< 0.
\end{aligned}$$

- On a closed orbit, we would have both

$$\oint \frac{d}{dt}\left(x^2 + y^2\right)\, dt < 0 \quad \text{and} \quad \oint \frac{d}{dt}\left(x^2 + y^2\right)\, dt = 0,$$

which is a contradiction.

2. Consider the system

$$\dot{x} = x\left[x(1-x) - y\right], \qquad \dot{y} = y(x-\mu),$$

where μ is a parameter.
(a) Find the fixed points and determine their linearized stability.
(b) Find the bifurcation points and classify the bifurcations.

Solution.

- (a) The fixed points for (x, y) are

$$(0,0), \qquad (1,0), \qquad (\mu, \mu(1-\mu))\,.$$

- The Jacobian matrix is

$$\begin{pmatrix} 2x - 3x^2 - y & -x \\ y & x - \mu \end{pmatrix}.$$

- The Jacobian matrix at $(0,0)$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & -\mu \end{pmatrix}$$

  with eigenvalues $\lambda = 0, -\mu$ (a degenerate case with a zero eigenvalue of multiplicity 2 at $\mu = 0$). The fixed point is unstable if $\mu < 0$ and linearly stable if $\mu > 0$.

- The Jacobian matrix at $(1,0)$ is

$$\begin{pmatrix} -1 & -1 \\ 0 & 1-\mu \end{pmatrix}$$

  with eigenvalues $\lambda = -1, 1-\mu$. The fixed point is unstable (saddle point) if $\mu < 1$ and asymptotically stable (stable node) if $\mu > 1$.

- The Jacobian matrix at $(\mu, \mu(1-\mu))$ is

$$\begin{pmatrix} \mu(1-2\mu) & -\mu \\ \mu(1-\mu) & 0 \end{pmatrix}$$

with trace $\tau = \mu(1-2\mu)$, determinant $\Delta = \mu^2(1-\mu)$, and eigenvalues

$$\lambda = \frac{1}{2}\mu\left[1 - 2\mu \pm \sqrt{4\mu^2 - 3}\right].$$

The equilibrium is asymptotically stable when $\tau < 0$ and $\Delta > 0$, which occurs when $\mu < 0$ or $1/2 < \mu < 1$. It is unstable when $0 < \mu < 1/2$ or $\mu > 1$.

- We're not asked to classify the fixed points, but one finds the following classification for this fixed point. Let $a = \sqrt{3/4}$. Then:

$$\begin{array}{rl} \mu \le -a & \text{stable node;} \\ -a < \mu < 0 & \text{stable spiral;} \\ 0 < \mu < 1/2 & \text{unstable spiral;} \\ 1/2 < \mu < a & \text{stable spiral;} \\ a \le \mu < 1 & \text{stable node;} \\ \mu > 1 & \text{saddle point.} \end{array}$$

- (b) The stability of the fixed points changes at $\mu = 0, 1/2, 1$.
- There is a (degenerate) transcritical bifurcation at $(x, y; \mu) = (0, 0; 0)$, and a transcritical bifurcation at $(x, y; \mu) = (1, 0; 1)$.
- There is a Hopf bifurcation at $(x, y; \mu) = (1/2, 1/4; 1/2)$, where the eigenvalues λ of the linearized system are a complex-conjugate pair that crosses the imaginary λ-axis (from the right-half to the left-half of the complex plane) as μ increases through $1/2$.

3. (a) Write the scalar ODE

$$\ddot{x} + \mu\left(x^2 - 4\right)\dot{x} + x = 1$$

as a first order system for (x, y) where

$$y = \frac{\dot{x}}{\mu} + \frac{x^3}{3} - 4x.$$

(b) Give a qualitative argument for the existence of a limit cycle solution when μ is large and positive. Sketch the limit cycle in the phase plane.

(c) Extra credit: Estimate the period of the limit cycle for large values of μ.

Solution.

- (a) The ODE can be written as $\mu\dot{y} + x = 1$, so

$$\dot{x} = \mu\left(y - \frac{x^3}{3} + 4x\right), \qquad \dot{y} = \frac{1-x}{\mu}.$$

  (b) For large μ, the trajectory of the limit cycle moves slowly along the curve

$$y = \frac{x^3}{3} - 4x \tag{1}$$

  and x jumps rapidly across the curve from the maximum at $(-2, 16/3)$ to $(4, 16/3)$ and from the minimum at $(2, -16/3)$ to $(-4, -16/3)$. Furthermore, if $y > \frac{x^3}{3} - 4x$, then x increases rapidly and if $y < \frac{x^3}{3} - 4x$, then x decreases rapidly. In either case, trajectories approach the slow curve and the loop described above, which suggests that the system has a stable limit cycle.

- (c) The period T of the limit cycle is given approximately by the time spent by the trajectory on the slow curve. If $t = 0$ at $(4, 16/3)$ and $t = T/2$ at $(2, -16/3)$, then

$$T = \int_0^{T/2} dt = \int_4^2 \frac{dx}{\dot{x}} \tag{2}$$

- For a trajectory close to the slow curve (1), we have approximately that

$$\dot{y} = \left(x^2 - 4\right)\dot{x},$$

and $\dot{y} = (1 - x)/\mu$ from the ODE, so for large μ we can use the approximation

$$\dot{x} = \frac{1}{\mu}\left(\frac{1 - x}{x^2 - 4}\right)$$

- Using this expressing in (2) and changing the order of the limits, we get that

$$\begin{aligned} T &= \mu \int_2^4 \frac{x^2 - 4}{x - 1}\, dx \\ &= \mu \int_2^4 \left(x + 1 - \frac{3}{x - 1}\right) dx \\ &= \mu \left[\frac{1}{2}x^2 + x - 3\log|x - 1|\right]_2^4 \\ &= \mu\left(8 - 3\log 3\right). \end{aligned}$$

4. Consider the nonlinear system

$$\dot{x} = y - x\left\{\left(x^2+y^2\right)^4 - \mu\left[\left(x^2+y^2\right)^2 - 1\right] - 1\right\}$$
$$\dot{y} = -x - y\left\{\left(x^2+y^2\right)^4 - \mu\left[\left(x^2+y^2\right)^2 - 1\right] - 1\right\}$$

(a) Write the system in polar coordinates (r, θ).
(b) State the Poincaré-Bendixson theorem.
(c) For $0 \leq \mu < 1$, show that $1/2 < r < 2$ is a trapping region, and deduce that it contains a limit cycle.
(d) Show that a Hopf bifurcation occurs at $\mu = 1$. Is it subcritical or supercritical?

Solution.

- (a) since $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$, we have

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}, \qquad \dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}.$$

  It follows that

$$\dot{r} = r\left\{1 - \mu + \mu r^4 - r^8\right\}, \qquad \dot{\theta} = -1.$$

- (b) **Poincaré-Bendixson Theorem**: Suppose that $R \subset \mathbb{R}^2$ is a closed, bounded subset of the plane. If a smooth, planar dynamical system has no fixed points in R and there is a trajectory $\mathbf{x}(t)$ that enters R and remains in R for all subsequent times t, then R contains a closed orbit. Moreover, the trajectory $\mathbf{x}(t)$ either is a closed orbit or it approaches a closed orbit as $t \to \infty$.[1]

- (c) Suppose $0 \leq \mu < 1$. If $0 < r < 1$, then $r^4 - 1 < 0$ and

$$1 + \mu\left(r^4 - 1\right) - r^8 > 1 + \left(r^4 - 1\right) - r^8 = r^4\left(1 - r^4\right) > 0.$$

  Similarly, if $r > 1$, then

$$1 + \mu\left(r^4 - 1\right) - r^8 < 1 + \left(r^4 - 1\right) - r^8 = r^4\left(1 - r^4\right) < 0.$$

  It follows that $r(t)$ is an increasing function of t when $r = 1/2$ and a decreasing function of t when $r = 2$, so trajectories that enter the annulus $1/2 \leq r \leq 2$ remain in the annulus for all subsequent times.

[1] Optional note: More precisely, the ω-limit set of $\mathbf{x}(t)$ is a closed orbit.

- The only equilibrium of the system is at $r = 0$, so the Poincaré-Bendixson theorem implies that the annulus $1/2 \leq r \leq 1$ contains a closed orbit. Moreover, since $r(t)$ is strictly increasing (decreasing) when $r = 1/2$ ($r = 2$), a trajectory that enters the annulus at the circle $r = 1/2$ ($r = 2$) cannot return to the circle, so it cannot be closed, and the trajectory must approach a limit cycle in the annulus as $t \to \infty$.

- The radial equation can be written as

$$\dot{r} = r\left(1 - r^4\right)\left(1 - \mu + r^4\right).$$

  It has fixed points: at $r = 0$ (unstable if $\mu < 1$, stable if $\mu > 1$); $r = 1$ (stable if $\mu < 2$, unstable if $\mu > 2$); and $r = (\mu - 1)^{1/4}$ for $\mu > 1$ (unstable if $1 < \mu < 2$, stable if $\mu > 2$).

- It follows that an unstable limit cycle bifurcates from the fixed point $r = 0$ at $\mu = 1$ into $\mu > 1$, where the fixed point is stable.

- If we classify the bifurcation by the stability of the limit cycle, this Hopf bifurcation is subcritical. (An unstable limits cycle shrinks to a stable fixed point which loses stability as μ decreases through 1, with a "hard" loss of stability for $\mu < 1$ in which the system jumps to a distant stable limit cycle at $r = 1$.)