Additional Problems: Set 3.

1. A planar Hamiltonian system is a system of the form

$$\dot{x} = \frac{\partial H}{\partial y}, \qquad \dot{y} = -\frac{\partial H}{\partial x}$$

where $H(x, y)$ is a smooth function of $(x, y) \in \mathbb{R}^2$. Show that

$$H(x, y) = \text{constant}$$

on trajectories and trajectories are tangent to the level curves of H.

2. Suppose that

$$H(x, y) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}y^2. \tag{1}$$

Sketch the phase plane of the corresponding Hamiltonian system. Compare the result with the phase plane of the gradient system

$$\dot{x} = -\frac{\partial V}{\partial x}, \qquad \dot{y} = -\frac{\partial V}{\partial y}$$

with $V(x, y) = H(x, y)$ that we discussed in class.

3. Consider the system

$$\dot{x} = \frac{\partial H}{\partial y} - \mu H \frac{\partial H}{\partial x} \qquad \dot{y} = -\frac{\partial H}{\partial x} - \mu H \frac{\partial H}{\partial y}$$

where $H(x, y)$ is defined in (1) and $\mu > 0$ is a small positive constant.

(a) Show that $\dot{H} < 0$ if $H > 0$ and $\dot{H} > 0$ if $H < 0$.

(b) Sketch the phase plane of the system.

(c) Describe all ω-limit sets of the system, and for each ω-limit set identify the points with that ω-limit set.