REAL ANALYSIS
Math 125A, Fall 2012
Final Solutions

1. (a) Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous on the closed, bounded interval $[0, 1]$ and $f(x) > 0$ for every $0 \leq x \leq 1$. Prove that the reciprocal function $1/f : [0, 1] \to \mathbb{R}$ is bounded on $[0, 1]$.
(b) Does this result remain true if: (i) $f : [0, 1] \to \mathbb{R}$ is not continuous on $[0, 1]$; (ii) $f : (0, 1) \to \mathbb{R}$ is continuous on the open interval $(0, 1)$?

Solution.

- (a) Let
$$m = \inf_{x \in [0,1]} f(x).$$
Since $f > 0$ on $[0, 1]$, we have $m \geq 0$. Since f is a continuous function on a compact set, it attains its infimum at some point in $[0, 1]$, which implies that $m > 0$. Therefore $f \geq m > 0$ and $0 < 1/f \leq 1/m$ is bounded on $[0, 1]$.

- (b) The result does not remain true in either case, since f need not attain its infimum. A counter-example for (i) is
$$f(x) = \begin{cases} x & \text{if } 0 < x \leq 1, \\ 1 & \text{if } x = 0. \end{cases}$$
A counter-example for (ii) is $f(x) = x$ for $0 < x < 1$.

2. (a) Define uniform continuity on $\mathbb{R}$ for a function $f : \mathbb{R} \to \mathbb{R}$.
(b) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous on $\mathbb{R}$. (i) Prove that $f + g$ is uniformly continuous on $\mathbb{R}$. (ii) Give an example to show that fg need not be uniformly continuous on $\mathbb{R}$.

Solution.

- (a) A function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in \mathbb{R}$ such that $|x - y| < \delta$.

- (b.i) Let $\epsilon > 0$. Choose $\delta_1 > 0$ such that
$$|f(x) - f(y)| < \frac{\epsilon}{2} \qquad \text{for all } x, y \in \mathbb{R} \text{ such that } |x - y| < \delta_1$$
and $\delta_2 > 0$ such that
$$|g(x) - g(y)| < \frac{\epsilon}{2} \qquad \text{for all } x, y \in \mathbb{R} \text{ such that } |x - y| < \delta_2.$$
Let $\delta = \min(\delta_1, \delta_2) > 0$. Then $|x - y| < \delta$ implies that
$$|(f + g)(x) - (f + g)(y)| \leq |f(x) - f(y)| + |g(x) - g(y)| < \epsilon,$$
which proves that $f + g$ is uniformly continuous on $\mathbb{R}$.

- (b.ii) An example is $f(x) = g(x) = x$. Then f, g are uniformly continuous on $\mathbb{R}$ (take $\delta = \epsilon$) but $(fg)(x) = x^2$ is not.

- To prove that x^2 is not uniformly continuous, let $\delta > 0$ and choose
$$x = \frac{1}{\delta} + \frac{\delta}{2}, \qquad y = \frac{1}{\delta}.$$
Then $|x - y| = \delta/2 < \delta$, but
$$|x^2 - y^2| = 1 + \frac{\delta^2}{4} > 1,$$
so the definition of uniform continuity fails for $\epsilon \geq 1$. (We can show similarly that it fails for all $\epsilon > 0$.)

3. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at zero and

$$f\left(\frac{1}{n}\right) = 0 \qquad \text{for all } n \in \mathbb{N}.$$

Prove that: (a) $f(0) = 0$; (b) $f'(0) = 0$.

Solution.

- (a) Since f is differentiable at 0, it is continuous at 0. The sequential definition of continuity then implies that

$$f(0) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0.$$

- (b) Since f is differentiable at 0, the limit

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x}$$

exists, so we can evaluate it on any sequence $x_n \to 0$. Using (a) and taking $x_n = 1/n$, we get that

$$f'(0) = \lim_{n\to\infty} nf\left(\frac{1}{n}\right) = 0.$$

4. Suppose that $f, g, h : \mathbb{R} \to \mathbb{R}$ are functions such that:
(a) $f(x) \leq g(x) \leq h(x)$ for all $x \in \mathbb{R}$, and $f(0) = h(0)$;
(b) f, h are differentiable at 0, and $f'(0) = h'(0)$.
Does it follow that g is differentiable at 0?

Solution.

- Yes, it does follow that g is differentiable at 0.

- Condition(a) implies that $f(0) = g(0) = h(0)$ and therefore also that

$$f(x) - f(0) \leq g(x) - g(0) \leq h(x) - h(0).$$

For $x > 0$, we have

$$\frac{f(x) - f(0)}{x} \leq \frac{g(x) - g(0)}{x} \leq \frac{h(x) - h(0)}{x},$$

and since

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = f'(0) = h'(0) = \lim_{x \to 0^+} \frac{h(x) - h(0)}{x},$$

the "sandwich" theorem implies that

$$\lim_{x \to 0^+} \frac{g(x) - g(0)}{x} = f'(0).$$

Similarly, for $x < 0$,

$$\frac{f(x) - f(0)}{x} \geq \frac{g(x) - g(0)}{x} \geq \frac{h(x) - h(0)}{x},$$

and the "sandwich" theorem implies that

$$\lim_{x \to 0^-} \frac{g(x) - g(0)}{x} = f'(0).$$

Since the left and right derivatives of g exist and are equal, it follows that g is differentiable at 0 and $f'(0) = g'(0) = h'(0)$.

5. (a) Determine the Taylor polynomial $P_n(x)$ of degree n centered at 0 for the function e^x.
(b) Give an expression for the remainder $R_n(x)$ in Taylor's theorem such that

$$e^x = P_n(x) + R_n(x).$$

(c) Prove that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$, with equality if and only if $x = 0$.
(d) Prove that $e^\pi > \pi^e$. HINT. Make a good choice of x in (c).

Solution.

- (a) The kth derivative of e^x is e^x, which is equal to 1 at $x = 0$, so the kth Taylor coefficient of $f(x) = e^x$ at zero is

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!},$$

  and

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n.$$

- (b) The expression for the Lagrange remainder is

$$R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) x^{n+1} = \frac{1}{(n+1)!} e^\xi \, x^{n+1}$$

  for some ξ strictly between 0 and x.

- (c) For $n = 1$, we get

$$e^x = 1 + x + \frac{1}{2} e^\xi x^2.$$

  Since $e^\xi > 0$, it follows that $e^x \geq 1 + x$, with equality if and only if $x = 0$.

- (d) Take

$$x = \frac{\pi}{e} - 1$$

  in the inequality from (c). This gives

$$e^{\pi/e - 1} > 1 + \frac{\pi}{e} - 1 \qquad \text{or} \qquad \frac{e^{\pi/e}}{e} > \frac{\pi}{e}.$$

  Multiply this inequality by e and take the eth power, to get $e^\pi > \pi^e$.

6. Suppose that (f_n) is a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$, and (x_n) is a sequence in $\mathbb{R}$ such that $x_n \to 0$ as $n \to \infty$. Prove or disprove the following statements.
(a) If $f_n \to f$ uniformly on $\mathbb{R}$, then $f_n(x_n) \to f(0)$ as $n \to \infty$.
(b) If $f_n \to f$ pointwise on $\mathbb{R}$, then $f_n(x_n) \to f(0)$ as $n \to \infty$.

Solution.

- (a) This statement is true. To prove it, we first observe that f is continuous since the uniform limit of continuous functions is continuous.

- Let $\epsilon > 0$ be given. We write

$$|f_n(x_n) - f(0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(0)|$$

  and estimate each of the terms of the right-hand side.

- Since $f_n \to f$ uniformly, there exists $N_1 \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \qquad \text{for all } x \in \mathbb{R} \text{ if } n > N_1.$$

- Since f is continuous at 0, there exists $\delta > 0$ such that

$$|f(x) - f(0)| < \frac{\epsilon}{2} \qquad \text{if } |x| < \delta,$$

  and since $x_n \to 0$ there exists $N_2 \in \mathbb{N}$ such that $|x_n| < \delta$ if $n > N_2$. Therefore

$$|f(x_n) - f(0)| < \frac{\epsilon}{2} \qquad \text{if } n > N_2,$$

- Let $N = \max(N_1, N_2)$. If $n > N$, then it follows that

$$|f_n(x_n) - f(0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

  which proves the result.

- (b) This statement is false. As a counter-example, let

$$f_n(x) = \begin{cases} 1 - n|x| & \text{if } |x| < 1/n, \\ 0 & \text{if } |x| \geq 1/n, \end{cases} \qquad x_n = \frac{2}{n}.$$

Then f_n is continuous and $f_n \to f$ pointwise, where

$$f(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$

Moreover $x_n \to 0$. However, we have $f_n(x_n) = 0$ for every n and $f(0) = 1$, so $f_n(x_n) \not\to f(0)$.

7. Consider the power series

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n x^{3n} = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \dots,$$

$$a_n = \frac{1}{2\cdot 3\cdot 5\cdot 6 \dots (3n-4)\cdot(3n-3)\cdot(3n-1)\cdot 3n}.$$

(a) For which $x \in \mathbb{R}$ does the series converge?
(b) Prove that $f''(x) = xf(x)$.

Solution.

- (a) We compute

$$\begin{aligned} r &= \lim_{n\to\infty} \left| \frac{a_{n+1}x^{3(n+1)}}{a_n x^{3n}} \right| \\ &= |x|^3 \lim_{n\to\infty} \frac{1}{(3n+2)(3n+3)} \\ &= 0. \end{aligned}$$

  The ratio test implies that the power series converges for every $x \in \mathbb{R}$. (Its radius of convergence is $R = \infty$.)

- (b) The differentiation theorem for power series implies that f is infinitely differentiable on $\mathbb{R}$, and its derivatives are the sum of the term-by-term differentiated power series. Moreover, the power series for the derivatives of f also converge on $\mathbb{R}$. Thus, using the identities

$$3\cdot 2a_1 = 1, \qquad 3n(3n-1)a_n = a_{n-1} \quad \text{for } n \geq 2,$$

  we get that

$$\begin{aligned} f''(x) &= \sum_{n=1}^{\infty} 3n(3n-1)a_n x^{3n-2} \\ &= x\left\{1 + \sum_{n=2}^{\infty} a_{n-1}x^{3n-3}\right\} \\ &= x\left\{1 + \sum_{n=1}^{\infty} a_n x^{3n}\right\} \\ &= xf(x) \end{aligned}$$

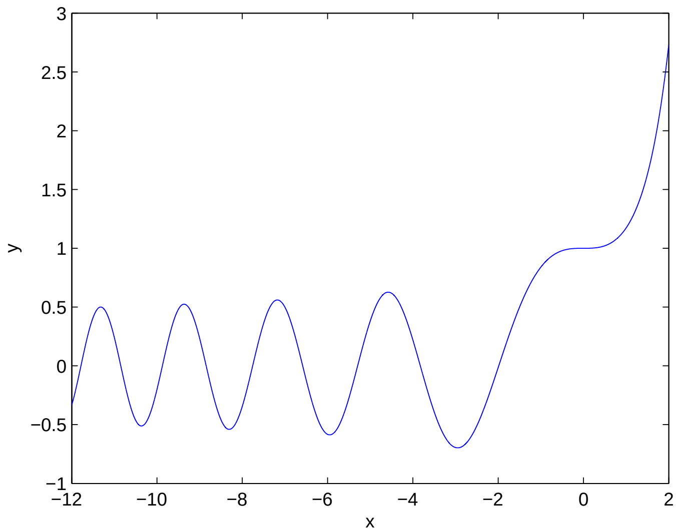

Figure 1: Graph of the Airy function $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^{3n}$.

Remark. Solutions of the ODE $y'' = xy$ are called Airy functions. For x large and positive, they behave like exponential functions, and for x large and negative, they behave like algebraically-decaying trigonometric functions.

The Airy functions describe a transition from oscillatory to exponential behavior, and can't be written in terms of elementary functions. They arise in many physical applications. In optics, they describe the electromagnetic field of a light wave near a boundary between light and shadow; and in quantum mechanics, they describe the wavefunction of a particle near a boundary between classically allowed and forbidden regions. (Here, the relevant Airy functions are the ones that decay as $x \to \infty$, rather than grow like f.)

The graph of the function in the problem is shown in Figure 1. It is worth noting that this graph was obtained by using MATLAB's built in routine for Airy functions, *not* by summing the power series explicitly. If x is large and negative, the power series consists of large terms with alternating signs, and these terms almost cancel. With a machine round-off error of approximately 2^{-16}, this cancelation makes the numerical values of the sum of the power series completely inaccurate when $x \lesssim -5$.

The moral is that a power series may not be as useful as it looks away from its central point if its rate of convergence becomes very slow.

8. (a) Define a metric on a set X.
(b) Consider the following functions defined for $x, y \in \mathbb{R}$ by:

$$d_1(x,y) = (x-y)^2, \qquad d_2(x,y) = |x^2 - y^2|, \qquad d_3(x,y) = |x - 2y|.$$

For each function, determine whether or not it is a metric on $\mathbb{R}$.

Solution.

- (a) A metric d on a set X is a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$:
  1. $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;
  2. $d(x,y) = d(y,x)$;
  3. $d(x,y) \leq d(x,z) + d(z,y)$.
- (b) None of these are metrics.
- The function d_1 is positive and symmetric, but it doesn't satisfy the triangle inequality e.g.

  $$d_1(1,0) + d_1(0,-1) = 1 + 1 = 2 < 4 = d_1(1,-1).$$

- The function d_2 fails only because $d_2(x,y) = 0$ implies that $x = \pm y$, not $x = y$ e.g. $d(1,-1) = 0$. However, d_2 satisfies the triangle inequality, since

  $$d_2(x,y) = \left|x^2 - z^2 + z^2 - y^2\right| \leq \left|x^2 - z^2\right| + \left|z^2 - y^2\right| = d_2(x,z) + d_2(z,y),$$

  and it would define a metric on $[0, \infty)$.
- Apart from being non-negative, the function d_3 is about as poor an excuse for a metric as you can find: it isn't symmetric; $d_3(x,y) = 0$ doesn't imply that $x = y$ e.g. $d(2,1) = 0$; and d_3 doesn't satisfy the triangle inequality e.g.

  $$d_3(4,2) + d_3(2,1) = 0 + 0 < 2 = d_3(4,1).$$