Real Analysis
Math 125A, Fall 2012
Solutions: Midterm 2

1. Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ and $f'(c) > 0$.
(a) Prove that there exists $\delta > 0$ such that $f(x) > f(c)$ for all $c < x < c + \delta$ and $f(x) < f(c)$ for all $c - \delta < x < c$.
(b) Does f have to be increasing in some neighborhood of c?

Solution.

- (a) Since the limit of the difference quotient
$$\lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right] = f'(c) > 0$$
is positive, there is a deleted neighborhood of c in which the difference quotient is positive.

- Explicitly, take $\epsilon = f'(c)/2 > 0$ and choose $\delta > 0$ such that
$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon \qquad \text{if } 0 < |x - c| < \delta.$$
Then, for $0 < |x - c| < \delta$,
$$\frac{f(x) - f(c)}{x - c} = f'(c) + \left[\frac{f(x) - f(c)}{x - c} - f'(c) \right] > f'(c) - \epsilon > 0.$$

- It follows that $f(x) - f(c) > 0$ if $0 < x - c < \delta$ and $f(x) - f(c) < 0$ is $-\delta < x - c < 0$, which proves the result.

- (b) No, f does not have to increasing in some neighborhood of c. For example, the function
$$f(x) = \begin{cases} x/2 + x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$
is differentiable, but not continuously differentiable, at 0 and
$$f'(0) = \lim_{x \to 0} \left[\frac{x/2 + x^2 \sin(1/x)}{x} \right] = \frac{1}{2} + \lim_{x \to 0} x \sin \frac{1}{x} = \frac{1}{2} > 0.$$

- However, f is not increasing in any neighborhood of 0. By the chain and product rule,

$$f'(x) = \frac{1}{2} - \cos\left(\frac{1}{x}\right) + 2x\sin\left(\frac{1}{x}\right)$$

  is continuous for $x \neq 0$ and takes negative values in every neighborhood of 0, at $x_n = 1/(2n\pi)$ for $n \in \mathbb{N}$ sufficiently large. Therefore, $f' < 0$ in some interval about x_n, and the monotonicity theorem implies that f is strictly decreasing in that interval.

2. Let (f_n) be a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ a function.
(a) Define: (i) $f_n \to f$ pointwise on $\mathbb{R}$; (ii) $f_n \to f$ uniformly on $\mathbb{R}$.
(b) Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is bounded for each $n \in \mathbb{N}$. (i) If $f_n \to f$ pointwise on $\mathbb{R}$ does f also have to be bounded? (ii) Prove that if $f_n \to f$ uniformly on $\mathbb{R}$, then f is bounded

Solution.

- (a.i) We have $f_n \to f$ pointwise on $\mathbb{R}$ if $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in \mathbb{R}$.

- (a.ii) We have $f_n \to f$ uniformly on $\mathbb{R}$ if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all } x \in \mathbb{R}.$$

- (b.i) The pointwise limit of bounded functions need not be bounded. For example, let

$$f_n(x) = \begin{cases} x & \text{if } |x| \le n, \\ 0 & \text{if } |x| > n. \end{cases}$$

Then $|f_n(x)| \le n$, so f_n is bounded. But $f_n(x) = x$ for all $n \ge |x|$, so $f_n \to f$ pointwise where $f(x) = x$, and f is not bounded on $\mathbb{R}$. (Note that $N = |x|$ gets arbitrarily large for large x; this is the non-uniform convergence.)

- (b.ii) Since $f_n \to f$ uniformly, there exists $N \in \mathbb{N}$ such that $n > N$ implies that

$$|f_n(x) - f(x)| < 1 \qquad \text{for all } x \in \mathbb{R}.$$

(Take $\epsilon = 1$ in the definition.) Choose any $n > N$. Since f_n is bounded, there is a constant M_n such that $|f_n(x)| \le M_n$ for all $x \in \mathbb{R}$. It follows that

$$|f(x)| \le |f(x) - f_n(x)| + |f_n(x)| < 1 + M_n \qquad \text{for all } x \in \mathbb{R},$$

which proves that f is bounded.

3. A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable if it is differentiable at every point of $\mathbb{R}$, and Lipschitz continuous if there is a constant $M \geq 0$ such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$.
(a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f' : \mathbb{R} \to \mathbb{R}$ is bounded. Prove that f is Lipschitz continuous.
(b) Give an example, with proof, of a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable but not Lipschitz continuous.
(c) Give an example, with proof, of a function $f : \mathbb{R} \to \mathbb{R}$ that is Lipschitz continuous but not differentiable.

Solution.

- (a) Since f is differentiable on $\mathbb{R}$, it is continuous on $\mathbb{R}$. Therefore, for every $x, y \in \mathbb{R}$ with $x < y$, say, we can apply the mean value theorem to f on the interval $[x, y]$ to get
$$\frac{f(x) - f(y)}{x - y} = f'(c)$$
for some $x < c < y$. Since f' is bounded, there is a constant $M \geq 0$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$ and it follows that
$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$
which proves that f is Lipschitz continuous. (The inequality is trivial if $x = y$.)

- (b) Let $f(x) = x^2$. Then f is differentiable on $\mathbb{R}$, but
$$\sup_{x \neq y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x - y|} = \sup_{x \neq y \in \mathbb{R}} \left| \frac{x^2 - y^2}{x - y} \right| = \sup_{x \neq y \in \mathbb{R}} |x + y| = \infty,$$
so f is not Lipschitz continuous on $\mathbb{R}$.

- (c) Let $f(x) = |x|$. Then the reverse triangle inequality
$$\big| |x| - |y| \big| \leq |x - y|$$
implies that f is Lipschitz continuous on $\mathbb{R}$ (with Lipschitz constant $M = 1$). On the other hand, f is not differentiable at 0 since
$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \operatorname{sgn} h$$
does not exist.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be the Thomae function

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ 1/q & \text{if } x = p/q \text{ is nonzero and rational.} \end{cases}$$

Here, if x is nonzero and rational, we write $x = p/q$ where the integers p, q have no common factors and $q > 0$ e.g. $f(-6/15) = f((-2)/5) = 1/5$.

Prove or disprove: f is differentiable at 0.

Solution.

- The Thomae function is not differentiable at 0.
- To show that the limit

$$f'(0) = \lim_{x \to 0} \left[\frac{f(x) - f(0)}{x} \right] = \lim_{x \to 0} \frac{f(x)}{x}$$

  does not exist, we consider sequences (x_n) and (y_n), where $x_n = 1/n$ and $y_n = \sqrt{2}/n$. Then $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$. Moreover, since y_n is irrational,

$$f(x_n) = \frac{1}{n}, \qquad f(y_n) = 0$$

- It follows that

$$\lim_{n \to \infty} \left[\frac{f(x_n) - f(0)}{x_n} \right] = \lim_{n \to \infty} \frac{1/n}{1/n} = 1, \qquad \lim_{n \to \infty} \left[\frac{f(y_n) - f(0)}{y_n} \right] = 0,$$

  and the sequential characterization of the limit implies that the limit defining $f'(0)$ does not exist.