Real Analysis
Math 125A, Fall 2012
Sample Final Questions

1. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{x^3}{1 + x^2}$$

Show that f is continuous on $\mathbb{R}$. Is f uniformly continuous on $\mathbb{R}$?

2. Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$ but $f'(x) \geq 1$ for all $x \neq 0$?

3. (a) Write out the Taylor polynomial $P_2(x)$ of order two at $x = 0$ for the function $\sqrt{1+x}$. and give an expression for the remainder $R_2(x)$ in Taylor's formula

$$\sqrt{1+x} = P_2(x) + R_2(x) \qquad -1 < x < \infty.$$

(b) Show that the limit

$$\lim_{x \to 0} \left[\frac{1 + x/2 - \sqrt{1+x}}{x^2} \right]$$

exists and find its value.

4. (a) Suppose $f_n : A \to \mathbb{R}$ is uniformly continuous on A for every $n \in \mathbb{N}$ and $f_n \to f$ uniformly on A. Prove that f is uniformly continuous on A.

(b) Does the result in (a) remain true if $f_n \to f$ pointwise instead of uniformly?

5. Define $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \frac{\sin(nx)}{1 + nx}.$$

(a) Show that f_n converges pointwise on $[0, \infty)$ and find the pointwise limit f.

(b) Show that $f_n \to f$ uniformly on $[a, \infty)$ for every $a > 0$.

(c) Show that f_n does not converge uniformly to f on $[0, \infty)$.

6. Suppose that

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}, \qquad g(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

(a) Prove that $f, g : \mathbb{R} \to \mathbb{R}$ are continuous.

(b) Prove that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f' = g$.

7. Let $P = \{2, 3, 5, 7, 11, \dots\}$ be the set of prime numbers.

(a) Find the radius of convergence R of the power series

$$\sum_{p \in P} x^p = x^2 + x^3 + x^5 + x^7 + x^{11} + \dots$$

(b) Show that

$$0 \le f(x) \le \frac{x^2}{1-x} \qquad \text{for all } 0 \le x < 1.$$

8. Let (X, d) be a metric space.

(a) Define the open ball $B_r(x)$ of radius $r > 0$ and center $x \in X$.

(b) Define an open set $A \subset X$.

(c) Show that the open ball $B_r(x) \subset X$ is an open set.