Sample Questions for Midterm 2
Math 125A, Fall 2012

1. For $\alpha \in \mathbb{R}$, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} |x|^\alpha \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Determine, with proof, for what values of α: (a) f is continuous at 0; (b) f is differentiable at 0; (c) f is continuously differentiable at 0.

2. (a) State the mean value theorem.
(b) If $\alpha > 1$, prove that

$$(1+x)^\alpha \geq 1 + \alpha x \qquad \text{for all } x > -1$$

with equality if and only if $x = 0$. (You can assume that $(x^\alpha)' = \alpha x^{\alpha-1}$ if $x > 0$ for every $\alpha \in \mathbb{R}$.)

3. Let I be an open interval containing 0. Suppose that the functions $f, g : I \to \mathbb{R}$ are differentiable at 0 with $f(0) = g(0) = 0$ and $g'(0) \neq 0$. Prove that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$

4. Prove or disprove the following converse to the Weierstrass M-test: If a series

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

of bounded functions $f_n : A \to \mathbb{R}$ converges absolutely and uniformly on A to a function $f : A \to \mathbb{R}$, then there exist constants $M_n \geq 0$ such that

$$|f_n(x)| \leq M_n \quad \text{for all } x \in A, \qquad \sum_{n=1}^{\infty} M_n < \infty.$$