Solutions to Sample Questions
Midterm 1: Math 125A, Fall 2012

1. (a) Suppose that $f : (0, 1) \to \mathbb{R}$ is uniformly continuous on $(0, 1)$. If (x_n) is a Cauchy sequence in $(0, 1)$ and $y_n = f(x_n)$, prove that (y_n) is a Cauchy sequence in $\mathbb{R}$.

(b) Give a counter-example to show that the result in (a) need not be true if $f : (0, 1) \to \mathbb{R}$ is only assumed to be continuous.

Solution.

- (a) Let $\epsilon > 0$ be given. Since f is uniformly continuous on $(0, 1)$, there exists $\delta > 0$ such that

  $$|x - y| < \delta \text{ and } x, y \in (0, 1) \text{ implies that } |f(x) - f(y)| < \epsilon.$$

  Since (x_n) is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that

  $$m, n > N \text{ implies that } |x_m - x_n| < \delta.$$

  It follows that

  $$m, n > N \text{ implies that } |f(x_m) - f(x_n)| < \epsilon,$$

  which shows that $(f(x_n))$ is a Cauchy sequence.

- (b) Suppose that $f(x) = 1/x$ for $x \in (0, 1)$ and $x_n = 1/n$ for $n \in \mathbb{N}$. Then f is continuous on $(0, 1)$ since it is a rational function with nonzero denominator. The sequence (x_n) is Cauchy since it converges to 0 and every convergent sequence is Cauchy (or give a direct proof). However, $y_n = f(x_n) = n$ and

  $$|y_n - y_m| \geq 1 \qquad \text{for every } m, n \in \mathbb{N} \text{ with } m \neq n,$$

  so (y_n) is not Cauchy.

Remark. Since every Cauchy sequence converges, it follows from this result that we can extend a uniformly continuous function $f : (0, 1) \to \mathbb{R}$ to a uniformly continuous function $\bar{f} : [0, 1] \to \mathbb{R}$ by defining

$$\bar{f}(0) = \lim_{n\to\infty} f(x_n)$$

where (x_n) is any sequence in $(0, 1)$ such that $x_n \to 0$ as $n \to \infty$, and

$$\bar{f}(1) = \lim_{n\to\infty} f(x_n)$$

where (x_n) is any sequence in $(0, 1)$ with $x_n \to 1$ as $n \to \infty$. (You have to check that the values $\bar{f}(0)$ and $\bar{f}(1)$ are independent of the choice of sequences to show that $\bar{f}$ is well defined.) However, we cannot extend a non-uniformly continuous function on $(0, 1)$, such as $f(x) = 1/x$, to a continuous function on $[0, 1]$.

2. (a) State the ϵ-δ definition for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at $c \in \mathbb{R}$.
(b) Define the floor function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \text{the largest integer } n \in \mathbb{Z} \text{ such that } n \leq x.$$

For example, $f(3.14) = 3$, $f(7) = 7$, $f(-3.14) = -4$. Determine, with proof, where f is continuous and where it is discontinuous.

Solution.

- (a) A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|x - c| < \delta \text{ implies that } |f(x) - f(c)| < \epsilon.$$

- (b) The floor function is discontinuous at every integer $c \in \mathbb{Z}$ and continuous at every $c \notin \mathbb{Z}$.

- If $c \in \mathbb{Z}$, define sequences (x_n), (y_n) by

$$x_n = c - \frac{1}{n}, \qquad y_n = c + \frac{1}{n}.$$

Then $x_n \to c$ and $y_n \to c$ as $n \to \infty$, but for every $n \in \mathbb{N}$

$$f(x_n) = c - 1, \qquad f(y_n) = c$$

so $f(x_n) \to c - 1$ and $f(y_n) \to c$ converge to different limits. The sequential definition of continuity implies that f is discontinuous at c. (It has a jump discontinuity at $c \in \mathbb{Z}$.)

- Suppose that $c \notin \mathbb{Z}$. Then $n < c < n + 1$ for some integer $n \in \mathbb{Z}$, and we can define $\delta > 0$ by

$$\delta = \min\left(c - n, n + 1 - c\right).$$

Since $|x - c| < \delta$ implies that $n < x < n + 1$ and $f(x) = n$ for all such x, we have

$$|x - c| < \delta \text{ implies that } |f(x) - f(c)| = 0.$$

Therefore we can use this $\delta > 0$ for every $\epsilon > 0$ in the definition of continuity, and f is continuous at c.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\lim_{x\to-\infty} f(x) = 0, \qquad \lim_{x\to\infty} f(x) = 0.$$

(a) Give a precise statement of what these limits mean.
(b) Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum value.
(c) Give examples to show that f may: (i) attain its maximum but not its infimum; (ii) attain both its maximum and minimum.

Solution.

- (a) The statement $\lim_{x\to-\infty} f(x) = 0$ means that for every $\epsilon > 0$ there exists $a \in \mathbb{R}$ (sufficiently negative) such that

  $$x < a \text{ implies that } |f(x)| < \epsilon,$$

  and $\lim_{x\to\infty} f(x) = 0$ means that for every $\epsilon > 0$ there exists $b \in \mathbb{R}$ (sufficiently positive) such that

  $$x > b \text{ implies that } |f(x)| < \epsilon.$$

- (b) If $f \equiv 0$ is identically zero, then the result follows immediately. If not, choose $c \in \mathbb{R}$ such that $f(c) \neq 0$. Taking $\epsilon = |f(c)| > 0$ in the limit definitions, we find that there exist $a, b \in \mathbb{R}$ such that

  $$|f(x)| < |f(c)| \qquad \text{for all } x < a \text{ and } x > b, \tag{1}$$

  where $a \leq c \leq b$ (since $f(x) \neq f(c)$ if $x < a$ or $x > b$).

- Since f is continuous on the compact interval $[a, b]$ it is bounded on $[a, b]$ and attains its maximum and minimum values on $[a, b]$. It follows from (1) that f is bounded on $\mathbb{R}$. Moreover, if $f(c) > 0$, then

  $$\max \{f(x) : x \in [a, b]\} \geq f(c)$$

  so f attains its global maximum on $\mathbb{R}$ at some point in $[a, b]$. Similarly, if $f(c) < 0$, then

  $$\min \{f(x) : x \in [a, b]\} \leq -f(c)$$

  so f attains its global minimum on $\mathbb{R}$ at some point in $[a, b]$.

- (c) The function, $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{1+x^2}$$

attains its maximum value, $f(0) = 1$, but not its infimum 0 on $\mathbb{R}$.

- The function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \frac{x}{1+x^2}.$$

attains both its maximum value, $g(1) = 1/2$, and minimum value, $g(-1) = -1/2$, on $\mathbb{R}$.