Sample Questions for Midterm 2: Solutions
Math 125A, Fall 2012

1. For $\alpha \in \mathbb{R}$, define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} |x|^\alpha \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Determine, with proof, for what values of α: (a) f is continuous at 0; (b) f is differentiable at 0; (c) f is continuously differentiable at 0.

Solution.

- (a) The function is continuous at 0 if and only if $\alpha > 0$. If $\alpha > 0$, then $|f(x)| \leq |x|^\alpha$ and $|x|^\alpha \to 0$ as $x \to 0$, so the "squeeze" theorem implies

$$\lim_{x \to 0} f(x) = 0.$$

  Since $f(0) = 0$, it follows that f is continuous at 0. On the other hand, if $\alpha \leq 0$, consider the sequence (x_n) defined by

$$x_n = \frac{1}{n\pi + \pi/2}.$$

  Then $x_n \to 0$ as $n \to \infty$ but

$$f(x_n) = |x_n|^\alpha \sin\left(n\pi + \frac{\pi}{2}\right) = (-1)^n |x_n|^\alpha$$

  does not converge. Therefore, by the sequential characterization of continuity, f is not continuous at 0.

- (b) The function is differentiable at 0 if and only if $\alpha > 1$. We have

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{|h|^\alpha \sin(1/h)}{h} \\ &= \lim_{h \to 0} \left[(\operatorname{sgn} h) |h|^{\alpha - 1} \sin\left(\frac{1}{h}\right) \right] \end{aligned}$$

  where $\operatorname{sgn} h = h/|h|$ for $h \neq 0$. As in (a), this limit exists if $\alpha - 1 > 0$, in which case $f'(0) = 0$, and it does not exist if $\alpha - 1 \leq 0$, in which case f is not differentiable at 0.

- (c) The function is continuously differentiable at 0 if $\alpha > 2$. By the product and chain rule, f is differentiable for $x \neq 0$ and

  $$f'(x) = \alpha(\operatorname{sgn} x)|x|^{\alpha-1} \sin\left(\frac{1}{x}\right) - |x|^{\alpha-2} \cos\left(\frac{1}{x}\right),$$

  where we use $(|x|^{\alpha})' = \alpha(\operatorname{sgn} x)|x|^{\alpha-1}$ for $x \neq 0$. As in (a),

  $$\lim_{x\to 0} f'(x) = 0$$

  if $\alpha - 2 > 0$, in which case f' is continuous at 0 since $f'(0) = 0$, and the limit does not exist if $\alpha - 2 \leq 0$, in which case f' is not continuous at 0.

2. (a) State the mean value theorem.

(b) If $\alpha > 1$, prove that

$$(1+x)^\alpha \geq 1 + \alpha x \qquad \text{for all } x > -1$$

with equality if and only if $x = 0$. (You can assume that $(x^\alpha)' = \alpha x^{\alpha-1}$ if $x > 0$ for every $\alpha \in \mathbb{R}$.)

Solution.

- (a) Mean value theorem. If $f : [a, b] \to \mathbb{R}$ is continuous on the closed, bounded interval $[a, b]$ and differentiable in the open interval (a, b), then there exists $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- (b) The function $f(x) = (1+x)^\alpha$ is differentiable, and therefore continuous, in $(-1, \infty)$, so we can apply the mean value theorem to f on the interval $[0, x]$ if $x > 0$, or $[x, 0]$ if $-1 < x < 0$. We find that there exists c between 0 and x such that

$$\alpha(1+c)^{\alpha-1} = \frac{(1+x)^\alpha - 1}{x},$$

  which implies that

$$(1+x)^\alpha = 1 + \alpha(1+c)^{\alpha-1}x.$$

- If $0 < c < x$, then $(1+c) > 1$ and $(1+c)^{\alpha-1} > 1$ since $\alpha > 1$. Therefore $(1+c)^{\alpha-1}x > x$ since $x > 0$ and

$$(1+x)^\alpha > 1 + \alpha x.$$

  If $-1 < x < c < 0$, then $(1+c)^{\alpha-1} < 1$. Therefore $(1+c)^{\alpha-1}x > x$ since $x < 0$ and

$$(1+x)^\alpha > 1 + \alpha x$$

  in this case also.

- If $x = 0$, we get equality, otherwise the inequality is strict.

- Alternatively, you can use Taylor's theorem to say that

$$(1+x)^\alpha = 1 + \alpha x + \frac{1}{2}\alpha(\alpha-1)\,(1+\xi)^{\alpha-2}\,x^2$$

for some ξ between 0 and x, and observe that the remainder is strictly positive if $-1 < x < \infty$ and $x \neq 0$.

3. Let I be an open interval containing 0. Suppose that the functions $f, g : I \to \mathbb{R}$ are differentiable at 0 with $f(0) = g(0) = 0$ and $g'(0) \neq 0$. Prove that

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)}.$$

Solution.

- By the linear approximation definition of the derivative, since f, g are differentiable at 0, there exist functions $r, s : I \to \mathbb{R}$ such that

$$f(x) = f(0) + f'(0)x + r(x) = f'(0)x + r(x), \qquad \lim_{x\to 0} \frac{r(x)}{x} = 0,$$
$$g(x) = g(0) + g'(0)x + s(x) = g'(0)x + s(x), \qquad \lim_{x\to 0} \frac{s(x)}{x} = 0.$$

- Since $f(0) = g(0) = 0$ and $g'(0) \neq 0$, it follows that

$$\begin{aligned}
\lim_{x\to 0} \frac{f(x)}{g(x)} &= \lim_{x\to 0} \frac{f'(0)x + r(x)}{g'(0)x + s(x)} \\
&= \lim_{x\to 0} \frac{f'(0) + r(x)/x}{g'(0) + s(x)/x} \\
&= \frac{\lim_{x\to 0}[f'(0) + r(x)/x]}{\lim_{x\to 0}[g'(0) + s(x)/x]} \\
&= \frac{f'(0)}{g'(0)}.
\end{aligned}$$

- An equivalent way to write this derivation is:

$$\begin{aligned}
\lim_{x\to 0} \frac{f(x)}{g(x)} &= \lim_{x\to 0} \frac{f(x) - f(0)}{g(x) - g(0)} \\
&= \lim_{x\to 0} \frac{\left[\frac{f(x)-f(0)}{x}\right]}{\left[\frac{g(x)-g(0)}{x}\right]} \\
&= \frac{\lim_{x\to 0}\left[\frac{f(x)-f(0)}{x}\right]}{\lim_{x\to 0}\left[\frac{g(x)-g(0)}{x}\right]} \\
&= \frac{f'(0)}{g'(0)}.
\end{aligned}$$

4. Prove or disprove the following converse to the Weierstrass M-test: If a series

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

of bounded functions $f_n : A \to \mathbb{R}$ converges absolutely and uniformly on A to a function $f : A \to \mathbb{R}$, then there exist constants $M_n \geq 0$ such that

$$|f_n(x)| \leq M_n \quad \text{for all } x \in A, \qquad \sum_{n=1}^{\infty} M_n < \infty.$$

Solution.

- The converse statement is false, and we give a counter-example. The idea is that the functions f_n can equal M_n at different points, so the sum of the functions at each point is strictly less than the sum of their bounds, and $\sum f_n$ may converge uniformly even though $\sum M_n$ diverges. In our example, at most one function is nonzero at each point.

- For $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1/n & \text{if } 1/2^n < x \leq 1/2^{n-1} \\ 0 & \text{otherwise} \end{cases}$$

  (Draw graphs of the first few f_n!)

- For every $x \in (0, 1]$, we have $f_n(x) \neq 0$ for only one $n \in \mathbb{N}$, namely the n for which $1/2^n < x \leq 1/2^{n-1}$, and $f_n(0) = 0$ for every $n \in \mathbb{N}$. Therefore the series

$$\sum_{n=1}^{\infty} f_n(x) = f(x)$$

  has at most one non-zero term for each x, so it converges pointwise and absolutely on $[0, 1]$ to the function

$$f(x) = \begin{cases} 1/n & \text{if } 1/2^n < x \leq 1/2^{n-1} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

- We claim that $\sum f_n$ converges uniformly to f on $[0,1]$. To prove this, we note that

$$f(x) - \sum_{k=1}^{n} f_k(x) = \begin{cases} 0 & \text{if } 1/2^n < x \leq 1 \\ 1/k & \text{if } 1/2^k < x \leq 1/2^{k-1} \text{ for some } k \geq n+1, \\ 0 & \text{if } x = 0. \end{cases}$$

It follows that

$$0 \leq f(x) - \sum_{k=1}^{n} f_k(x) \leq \frac{1}{n+1} \qquad \text{for all } x \in [0,1],$$

which proves the uniform convergence.

- Explicitly, if $\epsilon > 0$, let $N = 1/\epsilon$. (As required for uniform convergence, N is independent of x!) Then for every $n > N$ we have

$$\left| f(x) - \sum_{k=1}^{n} f_k(x) \right| < \frac{1}{N+1} < \epsilon \qquad \text{for all } x \in [0,1].$$

- Finally, note that for every $n \in \mathbb{N}$,

$$\sup_{x\in[0,1]} |f_n(x)| = \frac{1}{n},$$

so the smallest constant we can use in the bound $|f_n(x)| \leq M_n$ is $M_n = 1/n$, but the harmonic series $\sum 1/n$ diverges.

- Therefore, the series $\sum f_n$ converges uniformly and absolutely on $[0,1]$ and each f_n is bounded, but there do not exist constants M_n such that $|f_n(x)| \leq M_n$ on $[0,1]$ and $\sum M_n$ converges.

Remark. By changing the step functions in this example to triangular functions, we can make the f_n's continuous if we wish. Similar examples on $\mathbb{R}$ are a bit simpler e.g.

$$f_n(x) = \begin{cases} 1/n & \text{if } n-1 < x < n, \\ 0 & \text{otherwise,} \end{cases}$$

but as the previous example shows, even assuming that the functions are defined on a compact interval doesn't help.