MAT125B Lecture Notes

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Contents

1 Riemann integration

Throughout these notes the numbers $\epsilon > 0$ and $\delta > 0$ should be thought of as very small numbers. The objective of this section is to provide a rigorous definition for the integral $\int_a^b f(x)dx$ of a bounded function $f(x)$ on the interval $[a, b]$. We will see that the real number $\int_a^b f(x)dx$ is really the limit of sums of areas of rectangles.

1.1 Partitions and Riemann sums

Definition 1.1 (Partition P_δ of size $\delta > 0$)**.** *Given an interval $[a, b] \subset \mathbb{R}$, a partition P_δ denotes any finite ordered subset having the form*

$$P_\delta = \{a = x_0 < x_1 < \cdots x_{N-1} < x_N = b\} ,$$

where

$$\delta = \max\{x_i - x_{i-1} \mid i = 1, ..., N\}$$

denotes the maximum distance between any two adjacent partition point x_{i-1} and x_i, and where N denotes the number of subintervals that $[a, b]$ is partitioned into, with N depending on δ so that $N = N(\delta)$.

The simplest partitions have uniform spacing between partition points, in which case $\delta = \frac{b-a}{N}$ or conversely,

$$N(\delta) = \frac{b - a}{\delta} .$$

In order to build good intuition for partitions, there is no harm in considering our partitions to have uniform spacing between adjacent points, even though this is not the most general case.

Given a bounded function $f : [a,b] \to \mathbb{R}$, we are going to use a partition P_δ to define a piecewise constant approximation of the function $f(x)$ on the interval $[a,b]$. (The idea is that the integral of a piecewise constant function is the sum of the areas of a finite collection of rectangles.) We get to choose how we form this piecewise constant approximation to $f(x)$. The partition width $\delta > 0$ tells us how wide each rectangle will be, and what remains to be chosen is the height of each rectangle; this height depends on where we evaluate the function $f(x)$ over each subinterval $[x_{i-1}, x_i]$, and since there are an uncountably many points in the interval $[x_{i-1}, x_i]$, there are an uncountable many choices that we can make.

Definition 1.2 (Selection of evaluations points z_i)**.** *The evaluations points z_i are a collection of N points in the interval $[a,b]$ such that*

$$\{x_0 \le z_1 \le x_1 \le z_2 \le x_2 \le \cdots \le x_{N-1} \le z_N \le x_N\}\,.$$

Having a partition P_δ of the interval $[a,b]$ and having chosen the set of N evaluation points $z_1, z_2, ..., z_N$, we can now define the so-called Riemann sum.

Definition 1.3 (Riemann sum for the function $f(x)$)**.** *Given a function $f : [a,b] \to \mathbb{R}$, a partition P_δ, and a selection of evaluation points z_i, the Riemann sum of f is denoted by*

$$S_\delta(f) = \sum_{i=1}^{N} f(z_i)(x_i - x_{i-1})\,.$$

Again, note that since the choice of the evaluation points z_i is arbitrary, there are infinitely many Riemann sums associated with a single function and a partition P_δ.

Definition 1.4 (Integrability of the function $f(x)$)**.** *The function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if $S_\delta(f) \to S(f)$ as $\delta \to 0$. Equivalently, $f : [a,b] \to \mathbb{R}$ is Riemann integrable if for all $\epsilon > 0$, we can choose $\delta > 0$ sufficiently small so that*

$$|S_\delta(f) - S(f)| < \epsilon$$

for any Riemann sum $S_\delta(f)$ with maximum partition width δ.

Whenever the limit $S(f)$ exists we say that $S(f)$ is the integral of $f(x)$ over the interval $[a,b]$ and write

$$\int_a^b f(x)dx = S(f) = \lim_{\delta\to 0} S_\delta(f)\,.$$

Thus, $\int_a^b f(x)dx$ is just a limit of Riemann sums $S_\delta(f)$ whenever such a limit exists.

Definition 1.5 (Notation for integrable functions)**.** *We let*

$$\mathcal{R}(a,b) = \{f : [a,b] \to \mathbb{R} \mid f \textit{ is Riemann integrable}\}\,.$$

1.2 A criterion for integrability

Having provided a "rigorous" definition for the integral of a function, we must ask under what conditions is a function $f : [a, b] \to \mathbb{R}$ integrable. We will derive a few different criteria to ensure that $\int_a^b f(x)dx$ is a well-defined finite real number. We begin with a fairly general test.

Theorem 1.6 (Integrability criterion)**.** *Given $f : [a, b] \to \mathbb{R}$, if for every $\epsilon > 0$, there exists $\delta > 0$ sufficiently small so that*

$$|S^1_\delta(f) - S^2_\delta(f)| < \epsilon \tag{1.1}$$

for any two Riemman sums $S^1_\delta(f)$ and $S^2_\delta(f)$, then $f \in \mathcal{R}(a, b)$.

Proof. For simplicity, lets take $[a, b] = [0, 1]$, the unit interval. We take a sequence of Riemann sums $\{S_{1/n}(f)\}_{n=1}^\infty$ with each $S_{1/n}(f)$ having partition width $1/n$. Since $1/n \to 0$ as $n \to \infty$, we see that for any $\delta > 0$ given, we can choose N such that $1/N < \delta$, so that according to (1.1),

$$|S_{1/(n+m)}(f) - S_{1/n}(f)| < \epsilon \quad \forall n \geq N, m = 1, 2, 3, ...$$

This shows that $\{S_{1/n}(f)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $S_{1/n}(f) \to S(f)$ as $n \to \infty$. It follows that for any $\epsilon > 0$, we can choose $\bar{N} > 0$ sufficiently large so that

$$|S_{1/(n)}(f) - S(f)| < \epsilon \quad \forall n \geq \bar{N}\,. \tag{1.2}$$

It remains to show that the limit $S(f)$ of our Cauchy sequence is indeed the integral of f. In particular to show that $\int_a^b f(x)dx = S(f)$, we must show that for any $\bar{\epsilon} > 0$, we can choose $\delta > 0$ small enough so that

$$|S_\delta(f) - S(f)| < \bar{\epsilon}$$

for any Riemann sum $S_\delta(f)$. For this, we write

$$|S_\delta(f) - S(f)| = |S_\delta(f) - S_{1/n}(f) + S_{1/n}(f) - S(f)|\,,$$

and by the triangle inequality we see that

$$|S_\delta(f) - S(f)| \leq |S_\delta(f) - S_{1/n}(f)| + |S_{1/n}(f) - S(f)|\,.$$

If we choose n so large so that $n > \max(N, \bar{N})$ then we see that

$$|S_{1/n}(f) - S(f)| < \epsilon$$

by the inquality (1.2), and that

$$|S_\delta(f) - S_{1/n}(f)| < \epsilon$$

by the inequality (1.1) (we are using the $\delta > 0$ given in the statement of theorem). This shows that $|S_\delta(f) - S(f)| < 2\epsilon$, so we let $\bar{\epsilon} = 2\epsilon$ to complete the proof. $\square$

Theorem 1.6 is sometimes called the Cauchy criterion for integrability. The converse to Theorem 1.6 is much easier to establish.

Lemma 1.7. *If $f \in \mathcal{R}(a,b)$ then for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|S^1_\delta(f) - S^2_\delta(f)| < \epsilon$$

for any two Riemman sums $S^1_\delta(f)$ and $S^2_\delta(f)$.

Proof. Since $f \in \mathcal{R}(a,b)$, given $\epsilon > 0$, by Definition 1.4 we can choose $\delta > 0$ small enough so that

$$|S^1_\delta(f) - \int_a^b f(x)dx| < \frac{\epsilon}{2} \quad \text{and} \quad |S^2_\delta(f) - \int_a^b f(x)dx| < \frac{\epsilon}{2}\,.$$

Then, by the triangle inequality,

$$\begin{aligned}|S^1_\delta(f) - S^2_\delta(f)| &= |S^1_\delta(f) - \int_a^b f(x)dx + \int_a^b f(x)dx - S^2_\delta(f)| \\ &\le |S^1_\delta(f) - \int_a^b f(x)dx| + |S^2_\delta(f) - \int_a^b f(x)dx| < \epsilon\,.\end{aligned}$$

□

1.3 Upper and Lower Riemann Sums

The integrability criterion given by Theorem 1.6 is a bit too general, and we can refine it so as to make it more practical. This leads us to the notion of the upper and lower Riemann sum, known also as the upper and lower Darboux sum. The idea is to fix the selection points z_i so as to select two particular Riemann sums: the lower sum is based on the piecewise constant approximation to $f(x)$ whose graph lies just below the graph of $f(x)$, and the upper sum is based on the piecewise constant approximation to $f(x)$ whose graph lies just above the graph of $f(x)$.

Definition 1.8 (M_i and m_i)**.** *Given a partition P_δ, for $i = 1, ...N(\delta)$, we set*

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

and

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)\,.$$

Definition 1.9 (Upper and Lower Riemann (or Darboux) Sums)**.** *Given a partition P_δ, we let*

$$U_\delta(f) = \sum_{i=1}^N M_i(x_i - x_{i-1}) \quad \text{and } L_\delta \sum_{i=1}^N m_i(x_i - x_{i-1})$$

denote the upper and lower Riemann sums, respectively.

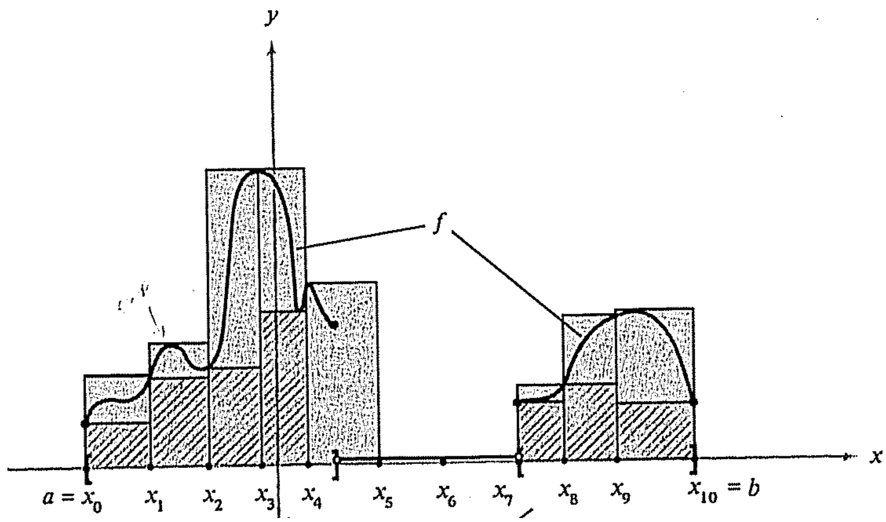


Figure 1: The shaded region is the area represents the upper Riemann sum, while the shaded area with diagonal lines represents the lower Riemann sum

Definition 1.10 (Upper and Lower Darboux Integral)**.** *We set*

$$U(f) = \inf_{\delta>0} U_\delta(f) \quad and \; L(f) = \sup_{\delta>0} L_\delta(f)\,.$$

$U(f)$ and $L(f)$ are sometimes called the upper and lower Darboux integrals, respectively.

We will see that the inf and the sup in this definition amount to passing to the limit as $\delta \to 0$. We will also see, that an equivalent criterion for integrability is the following:

Definition 1.11 (Integrability of $f(x)$ in terms of $L(f)$ and $U(f)$)**.** *$f \in \mathcal{R}(a,b)$ if $L(f) = U(f)$. In this case,*

$$\int_a^b f(x)dx = L(f) = U(f)\,.$$

While this definition may not look exactly the same as our original Definition 1.4, we will show that the two are indeed equivalent; on the other hand, Definition 1.11 is, in practice, much easier to compute with.

Example 1.12 (Compute $\int_0^1 x\,dx$)**.** *We subdivide $[0,1]$ into N subintervals, with partition width $\delta = 1/N$. It follows that for $i = 1, ..., N$, $m_i = x_{i-1} = (i-1)/N$ and $M_i = x_i = i/N$. Then,*

$$L_\delta(f) = \sum_{i=1}^{N} \frac{i-1}{N} \cdot \frac{1}{N} = \frac{1}{N^2} \sum_{i=0}^{N-1} i = \frac{(N-1)N}{2N^2}$$

and

$$U_\delta(f) = \sum_{i=1}^{N} \frac{i}{N} \cdot \frac{1}{N} = \frac{1}{N^2} \sum_{i=1}^{N} i = \frac{N(N+1)}{2N^2}\,.$$

Since $L(f) = \lim_{\delta\to 0} L_\delta(f) = \frac{1}{2}$ *and* $U(f) = \lim_{\delta\to 0} U_\delta(f) = \frac{1}{2}$*, we see that* $\int_0^1 x\,dx = L(f) = U(f) = \frac{1}{2}$*.*

(Note, that the computation in this example used the identity $\sum_{i=1}^{N} = \frac{1}{2}N(N+1)$. Also, note that the limit as $\delta \to 0$ is the same as the limit as $N \to \infty$ since $\delta = 1/N$.)

Example 1.13 (Compute $\int_0^1 x^2\,dx$)**.** *We use the same partition as the previous example; then, for* $i = 1, ..., N$*,* $m_i = x_{i-1}^2 = (i-1)^2/N^2$ *and* $M_i = x_i^2 = i^2/N^2$*, so that*

$$L_\delta(f) = \sum_{i=1}^{N} \frac{(i-1)^2}{N^2} \cdot \frac{1}{N} = \frac{1}{N^3} \sum_{i=0}^{N-1} i^2 = \frac{(N-1)N(2N-1)}{6N^3}$$

and

$$U_\delta(f) = \sum_{i=1}^{N} \frac{i^2}{N^2} \cdot \frac{1}{N} = \frac{1}{N^3} \sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6N^3}\,.$$

Passing to the limit as $\delta \to 0$*, we see that* $U(f) = L(f) = \frac{1}{3}$ *so that* $\int_0^1 x^2\,dx = \frac{1}{3}$*.*

Example 1.14 (A function $f(x)$ that is not (Riemann) integrable)**.** *Consider the function* $f(x)$ *on the unit interval* $[0,1]$ *given by*

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}\,.$$

Now, for any partition P_δ *of* $[0,1]$*, we see that each* $m_i = 0$ *and each* $M_i = 1$*. It follows that*

$$L_\delta(f) = \sum_{i=1}^{N} m_i(x_i - x_{i-1}) = 0 \text{ and } U_\delta(f) = \sum_{i=1}^{N} M_i(x_i - x_{i-1}) = 1\,,$$

so that $0 = L(f) \neq U(f) = 1$*, so that the Riemann integral does not exist.*

1.4 The refinement of a partition

Recall that a partition P_δ is a collection of points $\{x_0, x_1, ..., x_n\}$ with maximum width δ between any two adjacent points.

Suppose that $P^1_{\delta_1}$ and $P^2_{\delta_2}$ are two different partitions of the interval $[a,b]$. Then $P^1_{\delta_1} \cup P^2_{\delta_2}$ is usually a larger collections of points and usually has a smaller maximum partition width (the only way that this does not occur is if $P^1_{\delta_1} = P^2_{\delta_2}$). We can write this as

$$P^1_{\delta_1} \subset P^1_{\delta_1} \cup P^2_{\delta_2} \text{ and } P^2_{\delta_2} \subset P^1_{\delta_1} \cup P^2_{\delta_2}\,.$$

Definition 1.15 (Refinement of partitions)**.** *If $P^1_{\delta_1} \subset P^2_{\delta_2}$, then $P^2_{\delta_2}$ is a refinement of $P^1_{\delta_1}$ and $\delta_2 \le \delta_1$.*

Again, this notion becomes much easier to understand if we consider uniform partitions. For example, consider the unit interval $[0,1]$, and let

$$P^1_{\delta_1} = \{x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 = 1\}$$

so that $\delta_1 = 1/4$. Consider the refinement

$$P^2_{\delta_2} = \{x_0 = 0,\ x_1 = \frac{1}{8},\ x_2 = \frac{1}{4},\ x_3 = \frac{3}{8},\ x_4 = \frac{1}{2},\ x_5 = \frac{5}{8},\ x_6 = \frac{3}{4},\ x_7 = \frac{7}{8},\ x_8 = 1\}\,.$$

Notice that $\delta_2 = 1/8$ and that $P^1_{\delta_1} \subset P^2_{\delta_2}$.

1.5 Properties of upper and lower sums

The conceptual idea of the lower Riemann sum $L_\delta(f)$ is that we are *underestimating* the value of the integral $\int_a^b f(x)dx$, while with the upper Riemann sum $U_\delta(f)$ we are *overestimating* the value of the integral $\int_a^b f(x)dx$. In particular, if we fix some $\delta > 0$, then we expect that $L_\delta(f) \le \int_a^b f(x)dx$. As we make δ smaller and smaller, we hope that

$$L_\delta(f) \nearrow \int_a^b f(x)dx \text{ as } \delta \searrow 0\,.$$

Thus, we expect a certain monotonic behavior of the lower Riemann sums $L_\delta(f)$ as δ gets smaller and smaller. We can now make this idea precise.

Lemma 1.16. *If $f : [a,b] \to \mathbb{R}$ is bounded and $P^1_{\delta_1}$ and $P^2_{\delta_2}$ are two partitions of $[a,b]$ such that $P^1_{\delta_1} \subset P^2_{\delta_2}$, then*

$$L_{\delta_1}(f) \le L_{\delta_2}(f) \le U_{\delta_2}(f) \le U_{\delta_1}(f)\,. \tag{1.3}$$

Proof. Since $L_{\delta_2}(f)$ is formed using the m_i and $U_{\delta_2}(f)$ is formed using the M_i, then by definition the middle inequality in (1.3) holds. The first and third inequalities hold via the same argument, so we will provide it for the first inequality. In particular, we will show that

$$L_{\delta_1}(f) \le L_{\delta_2}(f) \quad \text{whenever} \quad P^1_{\delta_1} \subset P^2_{\delta_2}\,. \tag{1.4}$$

By an induction argument, it suffices to assume that $P^2_{\delta_2}$ has only one more point, x^*, than $P^1_{\delta_1}$. If

$$P^1_{\delta_1} = \{a = x_0 < x_1 < \cdots < x_N\}$$

then we can assume that

$$P^2_{\delta_2} = \{a = x_0 < x_1 < \cdots < x_{k-1} < x^* < x_k < \cdots < x_N\}$$

for some $k = 1, 2, ..., N$. In this case, the difference between $L_{\delta_1}(f)$ and $L_{\delta_2}(f)$ occurs only over the interval $[x_{k-1}, x_k]$, and we have that

$$\begin{aligned} L_{\delta_2}(f) - L_{\delta_1}(f) = & \inf_{x\in[x_{k-1},x^*]} f(x)\cdot(x^* - x_{k-1}) + \inf_{x\in[x^*,x_k]} f(x)\cdot(x_k - x^*) \\ & - \inf_{x\in[x_{k-1},x_k]} f(x)\cdot(x_k - x_{k-1})\,. \end{aligned} \tag{1.5}$$

To prove that (1.4) holds, we must show that the right-hand side of (1.5) is greater than or equal to zero. Since $P^1_{\delta_1} \subset P^2_{\delta_2}$, we know that $\inf P^2_{\delta_2} \le \inf P^1_{\delta_1}$ so that

$$\begin{aligned} \inf_{x\in[x_{k-1},x_k]} f(x)\cdot(x_k - x_{k-1}) &= \inf_{x\in[x_{k-1},x_k]} f(x)\cdot\Big((x_k - x^*) + (x^* - x_{k-1})\Big) \\ &\le \inf_{x\in[x_{k-1},x^*]} f(x)\cdot(x_k - x^*) + \inf_{x\in[x^*,x_k]} f(x)\cdot(x^* - x_{k-1})\,, \end{aligned}$$

which completes the proof. □

Lemma 1.17. *If $f : [a, b] \to \mathbb{R}$ is bounded, then for any two partitions $P^1_{\delta_1}$ and $P^2_{\delta_2}$,*

$$L_{\delta_1}(f) \le U_{\delta_2}(f)\,.$$

Proof. The set $P^1_{\delta_1} \cup P^2_{\delta_2}$ is also a partition of $[a, b]$ and since $P^1_{\delta_1} \cup P^2_{\delta_2} \subset P^1_{\delta_1}$ and $P^1_{\delta_1} \cup P^2_{\delta_2} \subset P^2_{\delta_2}$, Lemma 1.16 shows that

$$L_{\delta_1}(f) \le L_{\delta_{12}}(f) \le U_{\delta_{12}}(f) \le U_{\delta_2}(f)$$

where $L_{\delta_{12}}$ denotes any lower Riemann sum associated to the partition $P^1_{\delta_1} \cup P^2_{\delta_2}$, and $U_{\delta_{12}}$ denotes any upper Riemann sum associated to the partition $P^1_{\delta_1} \cup P^2_{\delta_2}$. □

Theorem 1.18. *If $f : [a, b] \to \mathbb{R}$ is bounded, then $L(f) \le U(f)$.*

Proof. If we fix an arbitrary partition $P_{\bar\delta}$ of $[a, b]$, then Lemma 1.17 shows that

$$L_{\bar\delta}(f) \le U_\delta(f) \quad \forall \delta > 0\,.$$

It follows that

$$L_{\bar\delta}(f) \le U(f)\,, \tag{1.6}$$

Since (1.6) holds for all partitions $P_{\bar\delta}$ with partition width $\bar\delta > 0$, then it follows that $L(f) \le U(f)$. □

Theorem 1.19 (Cauchy criterion for integrability in terms of upper and lower sums)**.** *A bounded function $f \in \mathcal{R}(a,b)$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that*

$$U_\delta(f) - L_\delta(f) < \epsilon\,. \tag{1.7}$$

Proof. Suppose that $\delta > 0$ is so small that (1.7) holds. Then we have that

$$\begin{aligned} U(f) \le U_\delta(f) &= U_\delta(f) - L_\delta(f) + L_\delta(f) \\ &\le \epsilon + L_\delta(f) \le \epsilon + L(f)\,. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we conclude that $U(f) \le L(f)$. Since Theorem 1.18 gives us the reverse inequality, $L(f) \le U(f)$, we must have that $L(f) = U(f)$ so that $f \in \mathcal{R}(a,b)$. □

1.6 Definition 1.4 implies Definition 1.11

Finally, let us show that if f is integrable according to our first definition (Definition 1.4) then it is also integrable according to our second definition (Definition 1.11).

Suppose that $f \in \mathcal{R}(a,b)$ in the sense of Definition 1.4, so that given any $\epsilon > 0$ there exists $\delta > 0$ so that

$$|S_\delta(f) - \int_a^b f(x)dx| < \epsilon\,.$$

For each $i = 1, ..., N(\delta)$, we select the evaluation points z_i so that

$$f(z_i) < m_i + \epsilon\,,$$

so that

$$S_\delta(f) \le L_\delta(f) + \epsilon(b-a)\,.$$

Hence,

$$L(f) \ge L_\delta(f) \ge S_\delta(f) - \epsilon(b-a) > \int_a^b f(x)dx - \epsilon - \epsilon(b-a)\,.$$

Since $\epsilon > 0$ is arbitrary, we see that $L(f) \ge \int_a^b f(x)dx$. A similar argument shows that $U(f) \le \int_a^b f(x)dx$, and since we know that $L(f) \le U(f)$, we must have that $L(f) = U(f) = \int_a^b f(x)dx$.

1.7 Monotonic and piecewise continuous functions are integrable

In this section, we will show that functions $f : [a,b] \to \mathbf{R}$ which are either piecewise monotonic or piecewise continuous are integrable.

Recall that $f : [a,b] \to \mathbb{R}$ is *increasing* if $f(x) \le f(y)$ whenever $x < y$ for any $x, y \in [a,b]$. Similarly, $f : [a,b] \to \mathbb{R}$ is *decreasing* if $f(x) \ge f(y)$ whenever $x < y$ for any $x, y \in [a,b]$. The function $f : [a,b] \to \mathbb{R}$ is called *monotonic* if it is either increasing or decreasing.

Theorem 1.20. *If $f : [a,b] \to \mathbb{R}$ is monotonic, then $f \in \mathcal{R}(a,b)$.*

Proof. We will prove the theorem for the case that f is increasing, as the argument for the case that f is decreasing is almost identitical.

Since f is increasing, by definition, $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$. It follows that f is bounded on $[a,b]$ by the value $f(b)$. In order to prove that $f \in \mathcal{R}(a,b)$, we will show that the Cauchy criterion of Theorem 1.19 is satisfied. To this end, for $\epsilon > 0$, we choose

$$\delta < \frac{\epsilon}{f(b) - f(a)}\,,$$

and choose any partition P_δ. Then

$$\begin{aligned}
U_\delta(f) - L_\delta(f) &= \sum_{i=1}^{N(\delta)} (M_i - m_i) \cdot (x_i - x_{i-1}) \\
&= \sum_{i=1}^{N(\delta)} (f(x_i) - f(x_{x-1}) \cdot (x_i - x_{i-1}) \\
&< \sum_{i=1}^{N(\delta)} (f(x_i) - f(x_{x-1}) \cdot \frac{\epsilon}{f(b) - f(a)} \\
&= \frac{\epsilon}{f(b) - f(a)} \sum_{i=1}^{N(\delta)} (f(x_i) - f(x_{x-1}) \\
&= \frac{\epsilon}{f(b) - f(a)} (f(b) - f(a)) = \epsilon\,.
\end{aligned}$$

□

Theorem 1.21. *If $f : [a,b] \to \mathbb{R}$ is continuous (denoted by $f \in C([a,b])$), then $f \in \mathcal{R}(a,b)$.*

Proof. Once again, we will use the Cauchy criterion of Theorem 1.19. Since $[a,b]$ is closed (hence compact), f is uniformly continuous on $[a,b]$; therefore, for any $\epsilon > 0$, we can choose $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{b-a} \quad \text{whenever} \quad x, y \in [a,b] \quad \text{and} \quad |x - y| < \delta\,. \tag{1.8}$$

Let P_δ denote any partition of $[a,b]$. Since f is continuous, we can replace the inf and sup with min and max, respectively, in the defintions of m_i and M_i, so that for each $i = 1, ..., N(\delta)$,

$$m_i = \min_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad M_i = \max_{x \in [x_{i-1}, x_i]} f(x)\,.$$

According the inequality (1.8), for each $i = 1, ..., N(\delta)$,

$$M_i - m_i \le \frac{\epsilon}{b-a}\,,$$

so that

$$\begin{aligned} U_\delta(f) - L_\delta(f) &< \sum_{i=1}^{N(\delta)} \frac{\epsilon}{b-a} \cdot (x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a} \sum_{i=1}^{N(\delta)} (x_i - x_{i-1}) \\ &= \frac{\epsilon}{b-a}(b-a) = \epsilon\,. \end{aligned}$$

□

1.8 The Riemann integral is linear

Since the Riemann integral is defined as the infinite limit of a sequence of finite sums, and as summation is linear operation, we expect that limiting integral should also be linear. This is indeed the case.

Theorem 1.22 (Linearity of the integral)**.** *Suppose that $f, g \in \mathcal{R}(a,b)$. Then*

(a) $f + g \in \mathcal{R}(a,b)$ and $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$;

(b) $\forall c \in \mathbb{R}$, $cf \in \mathcal{R}(a,b)$ and $\int_a^b cf(x)dx = c\int_a^b f(x)dx$.

Proof. We begin with the proof of part (a). Since f and g are in $\mathcal{R}(a,b)$, for any $\epsilon > 0$, we can choose $\delta_1 > 0$ (for f) and $\delta_2 > 0$ (for g) so that

$$\left| S_{\delta_1}(f) - \int_a^b f(x)dx \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| S_{\delta_2}(g) - \int_a^b f(x)dx \right| < \frac{\epsilon}{2}\,.$$

Let $\delta < \min(\delta_1, \delta_2)$ and let P_δ be any partition. It follows that

$$\left| S_{\delta}(f) - \int_a^b f(x)dx \right| < \frac{\epsilon}{2} \quad \text{and} \quad \left| S_{\delta}(g) - \int_a^b f(x)dx \right| < \frac{\epsilon}{2}\,.$$

We must show that the difference between $S_\delta(f+g)$ and $\int_a^b (f(x)dx + \int_a^b g(x)dx$ is less than ϵ whenever δ is sufficiently small. In other words, we wish to show that

$$\lim_{\delta \to 0} S_\delta(f+g) = \int_a^b (f(x)dx + \int_a^b g(x))dx\,. \tag{1.9}$$

Since by definition, $\int_a^b (f(x)+g(x))dx = \lim_{\delta\to 0} S_\delta(f+g)$, the proof will be complete when we establish (1.9). Notice that

$$\left|S_\delta(f+g) - [\int_a^b (f(x)dx + \int_a^b g(x)dx]\right| = \left|\left[S_\delta(f) - \int_a^b (f(x)dx\right] + \left[S_\delta(g) - \int_a^b g(x)dx\right]\right|$$
$$\leq \left|S_\delta(f) - \int_a^b (f(x)dx\right| + \left|S_\delta(g) - \int_a^b g(x)dx\right| < \epsilon\,,$$

which proves (1.9), and hence part (a).

To prove part (b), we can assume that $c \neq 0$, for otherwise the conclusion trivially holds. For $\epsilon > 0$, choose $\delta > 0$ sufficiently small so that for any partition P_δ, we have that

$$\left|S_\delta(f) - \int_a^b f(x)dx\right| < \frac{\epsilon}{|c|}\,.$$

It follows that

$$\left|S_\delta(cf) - c\int_a^b f(x)dx\right| = |c|\cdot\left|S_\delta(f) - \int_a^b f(x)dx\right| < |c|\cdot\frac{\epsilon}{|c|} = \epsilon\,.$$

□

Remark 1.23. *Note well, that in both parts of the above proof, we crucially relied on the linearity of $S_\delta(f)$; namely, we have used the fact that for any $\delta > 0$, $S_\delta(f+g) = S_\delta(f)+S_\delta(g)$ and that $cS_\delta(f) = S_\delta(cf)$.*

1.9 Further properties of the Riemann integral

Theorem 1.24. *If $f, g \in \mathcal{R}(a,b)$ and $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.*

Proof. We let $h = g - f$. According to Theorem 1.22, $h \in \mathcal{R}(a,b)$. Since $h \geq 0$, any lower Riemann sum $L_\delta(h) \geq 0$ for all $\delta > 0$. Hence, $\int_a^b h(x)dx \geq L(h) \geq 0$. Now, we apply Theorem 1.22 once again to find that

$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b h(x)dx \geq 0\,.$$

□

Theorem 1.25. *Suppose $f : [a,b] \to \mathbb{R}$. Let c be any point in (a,b). If $f \in \mathcal{R}(a,c)$ and $f \in \mathcal{R}(c,b)$, then $f \in \mathcal{R}(a,b)$ and*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx\,.$$

We leave the proof as an exercise for the student.

Definition 1.26 (Piecewise monotonicity)**.** *$f : [a,b] \to \mathbb{R}$ is piecewise monotonic if there exists a partition*

$$P_\delta = \{a = x_0, x_1, ..., x_{N-1}, x_N\}$$

such that f is monotonic on each subinterval $[x_{i-1}, x_i]$ for $i = 1, ..., N$. If the partition can be chose such that f is continuous on each subinterval $[x_{i-1}, x_i]$ for $i = 1, ..., N$, then f is piecewise continuous.

Theorem 1.27. *If $f : [a,b] \to \mathbb{R}$ is either piecewise continuous or a bounded piecewise monotonic function, then $f \in \mathcal{R}(a,b)$.*

In order to state our next theorem, we recall the *intermediate value theorem* for functions.

> *If $f \in C([a,b])$, then for every point y such that $f(a) < y < f(b)$, there exists at least one $x \in (a,b)$ such that $f(x) = y$.*

Theorem 1.28 (The intermediate value theorem for integrals)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous, then there exists at least one point $c \in [a,b]$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx \,.$$

Proof. Since f is continuous on $[a,b]$, a compact set, we let m and M denote the minimum and maximum of f, respectively. It follows that $m(b-a) \le \int_a^b f(x)dx \le M(b-a)$ and hence that

$$m \le \frac{1}{b-a} \int_a^b f(x)dx \le M \,. \tag{1.10}$$

Applying the intermediate value theorem for continuous functions to (1.10) yields the result. $\square$

Remark 1.29. *Theorem 1.28 states that whenever $f : [a,b] \to \mathbb{R}$ is continuous, there is always some point x in $[a,b]$ for which $f(x)$ is equal to average value of f over the entire interval $[a,b]$.*

Theorem 1.30. *If $f \in \mathcal{R}(a,b)$, then $|f| \in \mathcal{R}(a,b)$ and*

$$\left| \int_a^b f(x)dx \right| \le \int_a^b |f(x)|dx \,.$$

Proof. Let P_δ be any partition. Then for each $i = 1, ..., N(\delta)$,

$$\sup_{[x_{i-1},x_i]} |f|(x) - \inf_{[x_{i-1},x_i]} |f|(x) \le \sup_{[x_{i-1},x_i]} f(x) - \inf_{[x_{i-1},x_i]} f(x)$$

from which it follows that

$$U_\delta(|f|) - L_\delta(|f|) \le U_\delta(f) - L_\delta(f)\,.$$

This shows that whenever $f \in \mathcal{R}(a,b)$, then $|f| \in \mathcal{R}(a,b)$.

Next, since

$$-\int_a^b |f(x)|dx \le \int_a^b f(x)dx \le \int_a^b |f(x)|dx$$

then Theorem 1.24 finishes the proof. □

1.10 Interchanging limits with integrals

Theorem 1.31. *Let $\{f_n\}$ be a sequence of continuous functions on $[a,b]$, and suppose that $f_n \to f$ uniformly on $[a,b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx\,.$$

Proof. The limit f is continuous, so the functions $f_n - f$ are integrable on $[a,b]$. Since $f_n \to f$ uniformly, for $\epsilon > 0$, there exists N sufficiently large such that $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$ and $n > N$. Hence

$$\begin{aligned}\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| &= \left|\int_a^b [f_n(x) - f(x)]dx\right| \\ &\le \int_a^b |f_n(x) - f(x)|dx \le \int_a^b \frac{\epsilon}{b-a}dx = \epsilon\,,\end{aligned}$$

which proves the theorem. □

In fact, it is not necessary that the sequence of functions f_n be continuous on $[a,b]$.

Theorem 1.32. *Let $\{f_n\}$ be a sequence of bounded (Riemann) integrable functions defined on $[a,b]$. Suppose that $f_n \to f$ uniformly. Then f is integrable on $[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx\,.$$

There are at least two ways to prove this theorem. The first is to show that the uniform limit of integrable functions is integrable and then use the argument of Theorem 1.31. The second involves showing that the sequence of integrals $\int_a^b f_n(x)dx$ is a Cauchy sequence. The reader will be asked to supply a proof of Theorem 1.32 in the exercises using the second approach.

1.11 The fundamental theorem of calculus

Having characterized a large class of integrable functions, we return our focus to the actual computation of integrals. For integrals of 1-D intervals, the most practical approach relies on the use of so-called antiderivatives and the fundamental theorem of calculus. For a function F which is differentiable at x, we will denote $F'(x) = \frac{dF}{dx}(x)$.

Definition 1.33 (Antiderivative). *For $f : [a, b] \to \mathbb{R}$, the antiderivative of f is a continuous function $F : [a, b] \to \mathbb{R}$ such that F is differentiable on the open interval (a, b) and $F'(x) = f(x)$ for $x \in (a, b)$.*

Theorem 1.34 (Fundamental Theorem of Calculus). *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f has an antiderivative F, and*

$$F(b) - F(a) = \int_a^b f(x)dx\,. \tag{1.11}$$

If G is any other antiderivative of f, then the identity $G(b) - G(a) = \int_a^b f(x)dx$ also holds.

Remark 1.35. *According to Definition 1.33, since $F' = f$ on (a, b), we can equivalently write (1.11) as*

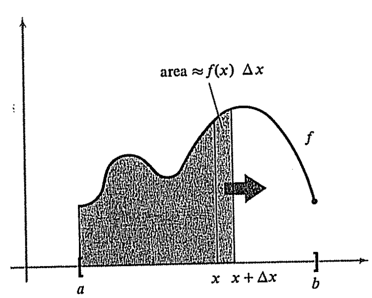
$$F(b) - F(a) = \int_a^b F'(x)dx\,. \tag{1.11'}$$

This shows that differentiation and integration are inverse operations, and in particular, we see that the integral of the derivative of a function is the function itself.

Remark 1.36. *To gain intuition for the fundamental theorem of calculus, let us suppose that $f(x) \geq 0$. We define*

$$F(x) = \int_a^x f(y)dy\,,$$

so that $F(x)$ represents the area under the graph of f from a to x. Recall that $F'(x) =$



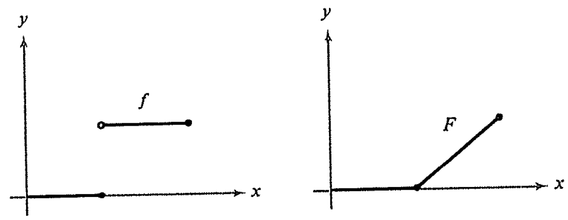
$\lim_{\Delta x \to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x}$. Now, the fact that $F'(x) = f(x)$ follows from looking at the figure: notice that $f(x)$ is the rate at which the area is increasing, since $F(x+\Delta x)-F(x) \sim f(x)\Delta x$.

Example 1.37. *$\int_0^{\pi/2} \sin x \, dx = 1$, since $\frac{d}{dx}(-\cos x) = \sin x$ and $-\cos(\pi/2) - (-\cos(0)) = 1$.*

An important question to ask is the following:

> Let $F(x) = \int_a^x f(t)dt$. Must F be differentiable if f is merely integrable (but not continuous)?

The answer is NO! Continuity in Theorem 1.34 is necessary. To see this, let



$$f(x) = \begin{cases} 0, & 0 \le x \le 1\,, \\ 1, & 1 < x \le 2\,. \end{cases}$$

and

$$F(x) = \begin{cases} 0, & 0 \le x \le 1\,, \\ x-1, & 1 < x \le 2\,. \end{cases}$$

We see that F is continuous, but not differentiable at $x = 1$.

Proof of Theorem 1.34 (Fundamental Theorem of Calculus). We define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(y)dy\,. \tag{1.12}$$

We begin by showing that F is the antiderivative of f. We let $x \in (a, b)$ an we choose $\delta > 0$ sufficiently small so that the open interval $(x, x + \delta) \subset (a, b)$. Next, we consider the following difference quotient:

$$\frac{F(x+\delta) - F(x)}{\delta} = \frac{\int_a^{x+h} f(y)dy - \int_a^x f(y)dy}{\delta} = \frac{1}{\delta}\int_x^{x+\delta} f(y)dy\,.$$

For $\epsilon > 0$, since f is continuous, we take $\delta > 0$ even small, if necessary, so that

$$|f(x) - f(y)| < \epsilon \text{ whenever } |x - y| < \delta\,.$$

It follows that

$$\left|\frac{1}{\delta}\int_x^{x+\delta} f(y)dy - f(x)\right| = \left|\int_x^{x+\delta}\frac{f(y)-f(x)}{\delta}dy\right|$$
$$\leq \int_x^{x+\delta}\frac{|f(y)-f(x)|}{\delta}dy < \frac{\epsilon\delta}{\delta} = \epsilon\,.$$

Thus, passing to the limit as $\delta \searrow 0$, we see that

$$f(x^+) = \lim_{\delta\searrow 0}\frac{F(x+\delta)-F(x)}{\delta}\,.$$

Similarly, we find that

$$f(x^-) = \lim_{\delta\searrow 0}\frac{F(x)-F(x-\delta)}{x-\delta}\,.$$

It follows that $F'(x)$ exists and that $F'(x) = f(x)$. But $F'(x) = \frac{d}{dx}\int_a^x f(y)dy$ by (1.12) so that

$$\frac{d}{dx}\left(F(x) - \int_a^x f(y)dy\right) = 0$$

Hence for some constant $c \in \mathbb{R}$,

$$F(x) - \int_a^x f(y)dy = c\,.$$

On the other hand, $\int_a^a f(y)dy = 0$ (since we are integrating the function over an interval of zero width) so that $F(a) = c$ and it follows that $F(x) - F(a) = \int_a^x f(y)dy$. Letting $x = b$, we obtain the claimed identity that

$$F(b) - F(a) = \int_a^b f(y)dy\,.$$

We must now show that F is also continuous $x = a$ and $x = b$. Since f is continuous on $[a, b]$ it has a maximum value, say M. Then

$$|F(a+\delta) - F(a)| \leq \int_a^{a+\delta} |f(y)|dy \leq \delta M\,,$$

which shows that F is continuous at $x = a$. A similar argument shows that F is also continuous at $x = b$.

Next, suppose that G is any antiderivative of f. Since $G'(x) = F'(x)$ and hence $\frac{d}{dx}(G - F) = 0$ on (a, b), we see that $G-F = c$ for some constant $c \in \mathbb{R}$. It follows that $G(b)-G(a) = \int_a^b f(y)dy$ for any antiderivative G. $\square$

Example 1.38. *If $f(x) = \frac{x^{p+1}}{p+1}$, then $f'(x) = x^p$ so by the Fundamental Theorem of Calculus, as long as $p \geq 0$,*

$$\int_a^b x^p dx = \frac{b^{p+1}}{p+1} - \frac{a^{p+1}}{p+1}\,.$$

In particular, $\int_0^1 x^2 dx = 1/3$, which is much easier than our previous computation of this integral in Example 1.13 which relied on the limit of Riemann sums.

An extremely useful corollary to the Fundamental Theorem of Calculus is the integration by parts formula.

Theorem 1.39 (Integration by parts)**.** *If f, g are continuous on $[a, b]$ and differentiable on (a, b), and if f', g' are integrable on $[a, b]$, then*

$$\int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a)\,.$$

Proof. Let $w = fg$. By the product rule, $w' = f'g + fg'$, and Exercise 1.3 shows that w' is integrable. By the Fundamental Theorem of Calculus,

$$\int_a^b w'(x)dx = w(b) - w(a) = f(b)g(b) - f(a)g(a)\,,$$

which completes the proof. □

1.12 Improper integrals

In previous sections, we have developed the theory of the Riemann integral for bounded functions $f(x)$ on a bounded interval $[a, b]$. We now develop the extension of this theory to unbounded functions or integrals over unbounded regions. This extension is known as the theory of *improper integrals*, which lead to convergence problems similar to those that are often encountered for infinite series.

It is customary to define the improper integral over an infinitely-long interval $[a, +\infty]$ as

$$\int_a^\infty f(x)dx = \lim_{L\to 0} \int_a^L f(x)dx\,,$$

and for unbounded functions near the $x = 0$ by

$$\int_0^b f(x) = \lim_{\epsilon\to 0} \int_\epsilon^b f(x)dx\,.$$

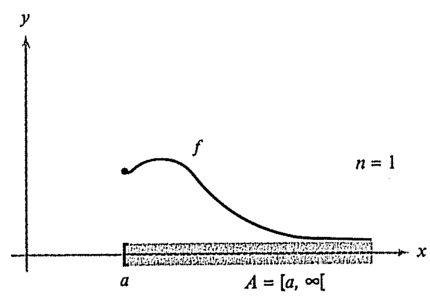
We must be extremely careful, however, when we try to extend such definitions to odd functions such as $\sin(x)$ over intervals that extend to both $-\infty$ and $+\infty$. In particular, we will *not* define

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{L\to\infty}\int_{-L}^{L} f(x)dx\,,$$

for in the case that $f(x) = x$, the above definition would lead to $\int_{-\infty}^{\infty} xdx = 0$. This is extremely problematic, as $\int_0^\infty xdx = \infty$ and $\int_{-\infty}^0 xdx = -\infty$, and in order to retain the linearity of the integral, we must be extremely careful. The solution, which avoids this issue of "cancellation of infinities," consists of the decomposition of the function $f(x)$ into its positive and negative parts. In the following, we shall explain this process in great detail.

1.12.1 Improper integrals over unbounded domains

We set $A = [a, \infty)$, and begin by studying functions functions $f(x) \geq 0$ for all $x \in A$. We then extend f to all of $\mathbb{R}$ by setting $f = 0$ on $(-\infty, a)$.



Definition 1.40. *Define $\int_A f dx := \int_a^\infty f(x)dx$ to be $\lim_{L\to\infty}\int_{-L}^{L} f(x)dx$ whenever this limit exists. Here f should be bounded and integrable on each closed interval $[-L, L]$. If $\int_a^\infty f(x)dx$ exists (and is finite), then we say that f is integrable.*

Theorem 1.41. *For $f \geq 0$, bounded and integrable on any closed interval $[-L, L]$, f is integrable on A if for any nested sequence of intervals $\{B_i\}_{i=1}^\infty$ such that (i) $B_i \subset B_{i+1}$, and (ii) for any closed interval C, we have that $C \subset B_i$ for i taken sufficiently large, then $\lim_{i\to\infty}\int_{B_i} f(x)dx$ exists. In this case, $\int_A f(x)dx = \lim_{i\to\infty}\int_{B_i} f(x)dx$.*

This theorem is checking if for $f \geq 0$, (a) f is Riemann integrable on any finite interval $[-L, L]$, and (b) if the limit of the integrals (as the interval length becomes larger and larger) is converging. When both (a) and (b) are verified, then the non-negative function f is called integrable over the unbounded interval A.

Note that if $f \geq 0$ is integrable, and $g : A \to \mathbb{R}$ is another functions such that $g \geq 0$ is Riemann integrable over any finite interval $[-L, L]$ and

$$0 \leq g \leq f\,,$$

then g is also integrable over the unbounded domain A. This follows since for any bounded interval $[-L, L]$, Theorem 1.24 asserts that

$$\int_{-L}^{L} g(x)dx \leq \int_{-L}^{L} f(x)dx\,,$$

and hence

$$\int_A g(x)dx = \lim_{L\to\infty}\int_{-L}^{L} g(x)dx \leq \lim_{L\to\infty}\int_{-L}^{L} f(x)dx = \int_A f(x)dx\,.$$

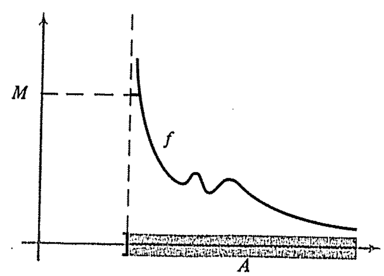
This is known as the *comparison test*.

1.12.2 Unbounded functions on possibly unbounded regions

We next consider the notion of integral for arbitrary functions $f \geq 0$ which are unbounded and defined on possibly unbounded intervals in $\mathbb{R}$.

Definition 1.42 (Cut-off for an unbounded function)**.** *For each $M > 0$, let*

$$f_M(x) = \begin{cases} f(x), & f(x) \leq M\,, \\ 0, & f(x) > M\,. \end{cases}$$

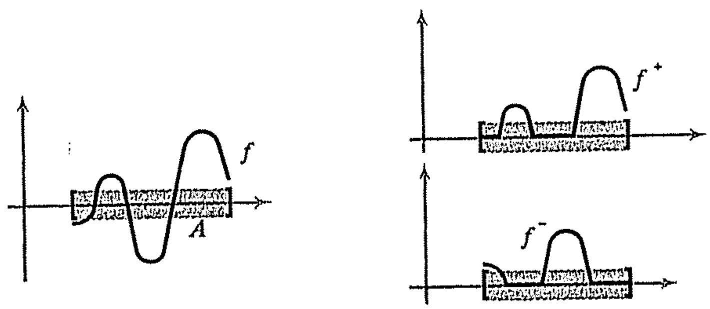


The function f_M is bounded by M from above, and bounded from below by zero, since we assumed that $f \geq 0$. Hence, we can define $\int_A f_M dx$ as in Definition 1.40. Note that $\int_A f_M dx$ increases as M increases and once again, $0 \leq f_M \leq f$. We then define

$$\int_A f(x)dx = \lim_{M\to\infty}\int_A f_M(x)dx$$

whenever this limit exists and is finite, in which case we say that f is *integrable* on A. Once again, we may invoke the *comparison test*: if $f \geq 0$ is integrable on the (possibly unbounded) interval A, and if $0 \leq g \leq f$, then g is also integrable on A. (Of course, we are again assuming that g is Riemann integrable over any finite interval $[-L, L]$.)

Having discussed the *improper integral* of functions $f \geq 0$, we now turn our attention to the general function $f : A \to \mathbb{R}$ which may also take negative values. We treat the case of the general function f by decomposing f into its positive and negative parts.



Definition 1.43 ($f = f^+ - f^-$)**.** *For a general function $f : A \to \mathbb{R}$, we set*

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0\,, \\ 0, & f(x) < 0\,. \end{cases} \quad \text{and} \quad f^-(x) = \begin{cases} -f(x), & f(x) \leq 0\,, \\ 0, & f(x) > 0\,. \end{cases}$$

Definition 1.44 (Integrability for general $f : A \to \mathbb{R}$)**.** *If both $\int_A f^+ dx$ and $\int_A f^- dx$ exist, we define*

$$\int_A f dx = \int_A f^+ dx - \int_A f^- dx,$$

and say that f is integrable on A.

Notice that not only is $f = f^+ - f^-$ (by definition), but that $|f| = f^+ + f^-$. Hence, f is integrable on A if $|f|$ is integrable on A since

$$\int_A f dx \leq |\int_A f dx| \leq \int_A f^+ dx + \int_A f^- dx = \int_A |f| dx\,.$$

Conversely, if $\int_A |f|$ exists, then both $\int_A f^+ dx$ and $\int_A f^- dx$ must be finite, since they are nonnegative and

$$0 \leq f^+ \leq |f| \quad \text{and} \quad 0 \leq f^- \leq |f|\,.$$

Thus, f is integrable on A iff $|f|$ is integrable on A.

This leads us to the notion of *absolute convergence* for an improper integral.

Definition 1.45 (Absolute convergence). *We say that $\int_A f dx$ is absolutely convergent if $\int_A |f| dx < \infty$.*

For integrals of functions defined over intervals in $\mathbb{R}$, i.e. for integrals in one space dimension, there is a more practical means of computing the *improper integral*, and the following theorem is in many ways more useful than Theorem 1.41. In particular, it is based on the Fundamental Theorem of Calculus.

Theorem 1.46 (Improper integrals in 1-D).

1. *(Unbounded interval) Suppose that $f : [a, \infty)$ is continuous and $f \geq 0$. Let F be an antiderivative of f. Then f is integrable on $[a, \infty)$ iff $\lim_{x\to\infty} F(x)$ exists. In this case*
$$\int_a^\infty f(x)dx = \left[\lim_{x\to\infty} F(x)\right] - F(a)\,.$$
2. *(Unbounded function near $x = a$) Suppose that $f : (a, b) \to \mathbb{R}$ is continuous and $f \geq 0$. Then f is integrable on $[a, b]$ iff*
$$\lim_{\epsilon\to 0^+} \int_{a+\epsilon}^b f(x)dx \text{ exists.}$$
This limit is equal to $\int_a^b f(x)dx$.

As with infinite series, we must be able to test an improper integral for convergence of divergence, and it should come as no surprise, that the tests closely resemble those used for infinite series. The *comparison test*, already discussed above, is one of the most useful tests. If $\int_a^\infty f(x)dx$ converges and $f \geq 0$ and $0 \leq g \leq f$, then $\int_a^\infty g(x)dx$ converges. As we have already discussed, $\int_a^\infty g(x)dx$ must converge since $\int_a^L g(x)dx$ increases as $L \to \infty$ and is bounded above by $\int_a^\infty f(x)dx$.

We have already defined absolute convergence of an integral. There is a weaker notion of convergence known as *conditional convergence* of the improper integral. Such conditional convergence is defined by
$$\int_a^\infty f(x)dx = \lim_{L\to\infty} \int_a^L f(x)dx$$
if the limit exists. For functions f which may take negative values, this is not the same as absolute convergence. For absolute convergence, the this limit must hold for both f^+ and f^-.

Example 1.47 (Absolute vs. conditional convergence). *Let $f(x) = \frac{\sin x}{x}$. We prove that $\int_1^\infty f(x)dx$ is conditionally but not absolutely convergent.*

If f were integrable on $[1,\infty)$, then $|f|$ would also be. But then

$$\int_1^\infty \frac{|\sin x|}{x} dx \ge \int_\pi^{n\pi} \frac{|\sin x|}{x} dx = \sum_{i=2}^n \int_{(i-1)\pi}^{i\pi} \frac{|\sin x|}{x} dx \ge \frac{2}{\pi} \sum_{i=2}^n \frac{1}{i}\,,$$

since on the interval $[(i-1)\pi\,,\,i\pi]$,

$$\frac{1}{x} \ge \frac{1}{i\pi}$$

and $\int_{(i-1)\pi}^{i\pi} |\sin x| dx = 2$. Then, since

$$\sum_{i=2}^n \frac{1}{i} \to \infty \text{ as } n \to \infty\,,$$

we see that $\int_1^\infty \frac{|\sin x|}{x} dx = +\infty$, so that $\int_1^\infty f(x)dx$ is not absolutely convergent.

On the other hand, $\lim_{L\to\infty} \int_1^L \frac{\sin x}{x} dx$ exists. To see this, note that

$$\int_1^L \frac{\sin x}{x} dx = -\int_1^L x^{-1} \frac{d}{dx} \cos x dx$$

so that integration by parts shows that

$$-\int_1^L x^{-1} \frac{d}{dx} \cos x dx = -\int_1^L x^{-2} \cos x dx + \left[-\frac{\cos L}{L} + \cos 1\right]\,,$$

and $\int_1^\infty \cos x/x^2 dx$ exists because

$$\int_1^L x^{-2} |\cos x| dx \le \int_1^L x^{-2} dx = 1 - \frac{1}{L}\,,$$

which converges as $L \to \infty$. It follows that $\int_1^\infty f(x)dx$ is conditionally convergent.

A driving theme throughout these lecture notes, has been the reduction of a difficult problem to an easier one, whose solution we know how to obtain. One of the fundamental difficulties in analysis is the determination of the (absolute) convergence of improper integrals. Most integrals do not have antiderivatives that can be expressed in closed-form (i.e., explicitly computed), but the comparison test offers a means of estimating these integrals, and obtaining upper bounds. In the following example, we provide a list of standard improper integrals which can be readily evaluated by either direct integration, successive integration by parts, or some other well-known calculus tricks. In practice, these integrals are used in conjunction with the comparison test for the purpose of determining the integrability of many large classes of functions.

Example 1.48 (List of well-known improper integrals)**.**

1. $\int_1^\infty x^p dx \begin{cases} \text{converges if } p < -1 \\ \text{diverges if } p \geq -1 \end{cases}$;

2. $\int_0^1 x^p dx \begin{cases} \text{converges if } p > -1 \\ \text{diverges if } p \leq -1 \end{cases}$;

3. $\int_1^\infty e^{-x} x^p dx$ *converges for all p;*

4. $\int_0^a e^{1/x} x^p dx$ *diverges for all p;*

5. $\int_0^a \log x \, dx$ *converges;*

6. $\int_1^\infty \left(\frac{1}{\log x}\right) dx$ *diverges.*

We begin by considering 1. Here we will apply Theorem 1.46. Notice that

$$\int_1^L x^p dx \begin{cases} \left.\frac{x^{p+1}}{p+1}\right|_1^L \,, & p \neq -1 \,, \\ \\ \log x|_1^L \,, & p = -1 \,. \end{cases}$$

Now $\log L \to \infty$ *as* $L \to \infty$*, and* $L^{p+1} \to \infty$ *as* $L \to \infty$ *if* $p+1 > 0$ *or* $p > -1$*. Part 2. is similar, and parts 5. and 6. can be proved in the same way also. The proof of parts 3. and 4. will be left as exercises, but we note that convergence of these integrals demonstrates the so-called Dirichlet test for integrals (generalizing the Dirichlet test for series) in which a function* f *which decays to zero is multiplied against a continuous function* g *that is integrable on all finite intervals; the result is that the product* fg *is integrable on the unbounded interval.*

Example 1.49. *The integral* $\int_1^\infty \frac{1}{\sqrt{x^3+1}} dx$ *converges.*

The integrand becomes unbounded as $x \to \infty$*. Notice that on our interval of interest,* $x \geq 1$ *and for such* x*, we see that*

$$\frac{1}{\sqrt{x^3+1}} \leq \frac{1}{\sqrt{x^3}} = x^{-3/2} \,,$$

and $\int_1^\infty x^{-3/2} dx$ *converges by part 1. of Example 2.51. Hence, the comparison principle shows that* $\int_1^\infty \frac{1}{\sqrt{x^3+1}} dx$ *converges as well.*

Proof of Theorem 1.46. Part 1. Suppose that $L > a$ and extend f to $[-L, \infty$ by setting $f = 0$ on $[-L, a)$. Then $\int_{-L}^L f(x)dx = \int_a^L f(x)dx$, and by the Fundamental Theorem of

Calculus, $\int_a^L f(x)dx = F(L) - F(a)$. Hence, $\lim_{L\to\infty} \int_{-L}^L f(x)dx$ exists iff $\lim_{L\to\infty} F(L)$ exists, and by definition $\int_a^f (x)dx = [\lim_{L\to\infty} F(L)] - F(a)$.

Part 2. For the second part, we need an approximate to f which cuts-off the singular behavior of f near $x = a$. To this end, we set

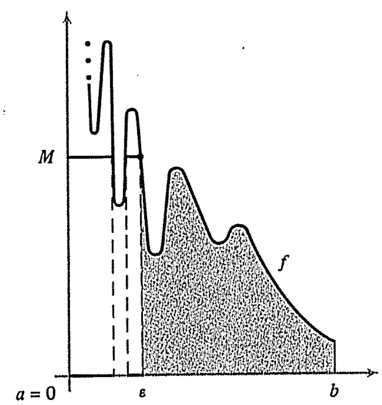
$$f^\epsilon(x) = \begin{cases} 0\,, & x \in [a, a+\epsilon]\,, \\ f(x)\,, & x \in (a+\epsilon, b]\,. \end{cases}$$

We proceed in two steps.

Step 1. First, recall the definition of the function f_M given in Definition 1.12.2. Then, with M given by $M = \sup_{x\in[a+\epsilon\,,\,b]} f(x)$, we see that $f_M(x) = f(x)$ if $f(x) \le M$ and $f_M = 0$ otherwise; hence,

$$\int_a^b f^\epsilon(x)dx = \int_{a+\epsilon}^b f(x)dx \le \int_a^b f_M(x)dx$$

since $f \le f_M$ on $[a+\epsilon\,,\,b]$. Note that $f_M(x)$ might not be zero on $[a\,,\,a+\epsilon]$ as in the figure.



Step 2. For any $\epsilon > 0$ and M,

$$\int_a^b f_M(x)dx - \int_a^b f^\epsilon(x)dx \le \epsilon M\,,$$

since $f_M \le M$ on $[a, a+\epsilon]$, and for $x \in (a+\epsilon, b]$, $f_M(x) = f(x)$, so that

$$\int_a^b f_M(x)dx - \int_a^b f^\epsilon(x)dx = \left(\int_a^{a+\epsilon} f_M(x)dx + \int_{a+\epsilon}^b f(x)dx\right) - \int_{a+\epsilon}^b f_M(x)dx \le \epsilon M\,,$$

as $f_M \le M$.

In order to prove the theorem, suppose that $\int_a^b f_M(x)dx \to I$ as $M \to \infty$. Notice that since $f \geq 0$, $\int_a^b f_M(x)dx$ increases as M increases. We remains to show that $\int_a^b f^\epsilon(x)dx$ also converges to I as $\epsilon \to 0$, but it clearly increases to a value which is bounded above by I according to Step 1. Hence, given $\delta > 0$, choose M such that $I - \int_a^b f_M(x)dx < \delta/2$. Then if we let $\epsilon = \delta/2M$, by Step 2., $\int_a^b f_M(x)dx - \int_a^b f^\epsilon(x)dx < \delta/2$. It follows that $I - \int_a^b f^\epsilon(x)dx < \delta$, so that $\lim_{\epsilon\to 0} \int_a^b f^\epsilon(x)dx = I$.

The converse follows in essentially the same way, again using Steps 1 and 2 to show that if $\int_a^b f^\epsilon(x)dx \to I$, then $\int_a^b f_M(x)dx \to I$. $\square$

1.13 Exercises

Problem 1.1. *Give an example of a function f on $[0,1]$ that is not integrable for which $|f|$ is integrable. (Hint: Modify Example 1.14.)*

Problem 1.2. *Let $f : [a,b] \to \mathbb{R}$ and suppose that $|f(x)| \leq A < \infty$ for all $x \in [a,b]$.*

(a) Show that $U_\delta(f^2) - L_\delta(f^2) \leq 2A\Big(U_\delta(f) - L_\delta(f)\Big)$ for all partitions P_δ. (Hint: Recall the identity $[(f(x) - f(y)]\,[f(x) + f(y)] = f(x)^2 - f(y)^2$.)

(b) Show that if $f \in \mathcal{R}(a,b)$, then $f^2 \in \mathcal{R}(a,b)$.

Problem 1.3. *Suppose that $f, g \in \mathcal{R}(a,b)$. Using the identity $(f-g)^2 - (f+g)^2 = 4fg$, show that $fg \in \mathcal{R}(a,b)$.*

Problem 1.4. *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous, $f(x) \geq 0$ for $x \in [a,b]$, and $f(x) > 0$ for some $x \in [a,b]$. Then $\int_a^b f(x)dx > 0$.*

Problem 1.5. *Let*

$$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} .$$

Show that $f \in \mathcal{R}(0,1)$.

Problem 1.6. *(a) Use integration by parts to evaluate*

$$\int_0^1 x \arctan x \, dx \, .$$

(Hint. If in the integration by parts formula, you set $f(x) = \arctan x$, then $f'(x) = \frac{1}{1+x^2}$.)

(b) In the integration by parts formula, if you used $g(x) = \frac{x^2}{2}$ in part (a), do the computation once again with $g(x) = \frac{x^2+1}{2}$ for an interesting surprise.

Problem 1.7. *Prove Theorem 1.32 using the following argument:*

1. *Prove that f is bounded.*
2. *With $I = \int_a^b f(x)dx$, we must find a number I such that for all $\epsilon > 0$, there is a $\delta > 0$ such that*
$$|S_\delta(f) - I| < \epsilon\,.$$
If such an I exists, we want to show that $I = \lim_{n\to\infty} \int_a^b f_n(x)dx$. Show that $\int_a^b f_n(x)dx$ is a Cauchy sequence.
3. *Use a triangle inequality argument to show that with $1/N < \delta$ for some integer N chosen sufficiently large,*
$$|S_{1/N}(f) - I| \le |S_{1/N}(f) - S_{1/N}(f_N)| + |S_{1/N}(f_N) - \int_a^b f_N(x)dx| + |\int_a^b f_N(x)dx - I|\,.$$
4. *Conclude that $|S_\delta(f) - I| < \epsilon$ and hence that $I = \int_a^b f(x)dx$.*

Problem 1.8. *Prove all of the assertions of Example 2.51.*

Problem 1.9. *Compute $\int_0^\infty \frac{1}{(1+x)^2}dx\,.$*

Problem 1.10. *Is $\int_0^\infty x^p dx$ convergent for any p? If so, for which p?*

2 Differentiable mappings of $\mathbb{R}^n$ to $\mathbb{R}^m$

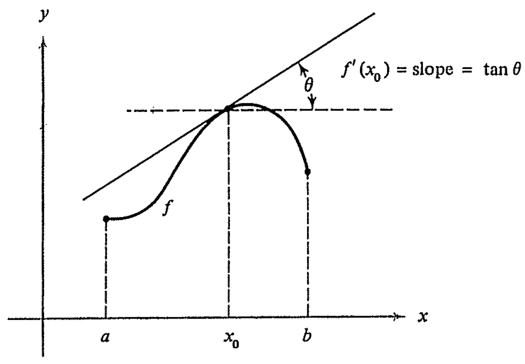
We consider functions $f : \mathbb{R}^n \to \mathbb{R}^m$, called mappings, and study their differentiability properties. A certain amount of lower-division linear algebra will be necessary, and the reader may wish to review linear transformations and their matrix representations.

2.1 The derivative $f : \mathbb{R}^n \to \mathbb{R}^m$

2.1.1 The one-dimensional case

We begin with a review of the derivative of a function defined over an interval in $\mathbb{R}$.

Definition 2.1 (Derivative in one-dimension)**.** *$f : (a, b) \to \mathbb{R}$ is differentiable at $x_0 \in (a, b)$ if for $h > 0$ chosen sufficiently small such that $x_0 + h \in (a, b)$, the limit*
$$\frac{df}{dx}(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$
exists. We often write $f'(x_0)$ for $\frac{df}{dx}(x_0)$, and call $f'(x_0)$ the derivative of f at x_0.



Notice that for functions over one-dimensional intervals, the derivative is the limit of a sequence of difference quotients, whenever this limit exists, with each difference quotient representing the approximate slope of the line tangent to the graph of the function f at the point x_0. The formula in Definition 2.1 can also be expressed as

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h} = 0\,. \tag{2.1}$$

The number $h > 0$ is the interval width over which the approximate slope of the function f is computed at x_0. We can define the point $x \in (a, b)$ to be

$$x = x_0 + h\,,$$

in which case (2.1) takes the form

$$\lim_{x\to x_0} \frac{|f(x) - f(x_0) - f'(x_0))(x - x_0)|}{|x - x_0|} = 0\,.$$

2.1.2 The multi-dimensional case

Let $A \subset \mathbb{R}^n$ denote an open subset. An element x of $\mathbb{R}^n$ is an n-vector, so that with respect to the usual basis of $\mathbb{R}^n$, given by $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, ..., 0)$, ..., $e_n = (0, 0, ..., 1)$,

$$x = (x_1, x_2, ..., x_n)\,.$$

For each $i = 1, ..., n$, x_i denotes the ith component of the n-vector x. A map $f : \mathbb{R}^n \to \mathbb{R}^m$ can also be written in terms of its components:

$$f(x) = (f_1(x), ..., f_m(x)) = (f_1(x_1, ..., x_n),\ .\ .\ .\ , f_m(x_1, ..., x_n))\,. \tag{2.2}$$

So, a map $f : \mathbb{R}^n \to \mathbb{R}^m$ has m-components and each component is a function of n independent variables $(x_1, ..., x_n)$.

Definition 2.2 (Derivative in multi-dimensions)**.** *A map* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is said to be differentiable at the point* $x \in A$ *if there exists a linear transformation, denoted* $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$ *and called the derivative of* f *at* x_0*, such that*

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)\|}{\|x - x_0\|} = 0\,.$$

We are using $\|\cdot\|$ to denote the Euclidean norm on $\mathbb{R}^m$, and the notation $Df(x_0)\cdot(x-x_0)$ denotes the value of the linear map $Df(x_0)$ applied to the vector $(x - x_0) \in \mathbb{R}^n$. As an example, fix $n = 2$ and $m = 1$ so that $f : \mathbb{R}^2 \to \mathbb{R}$. Then the matrix (or vector in this case) representation of $Df(x_0)$ (with respect to the standard orthogonal basis of $\mathbb{R}^2$) is given by

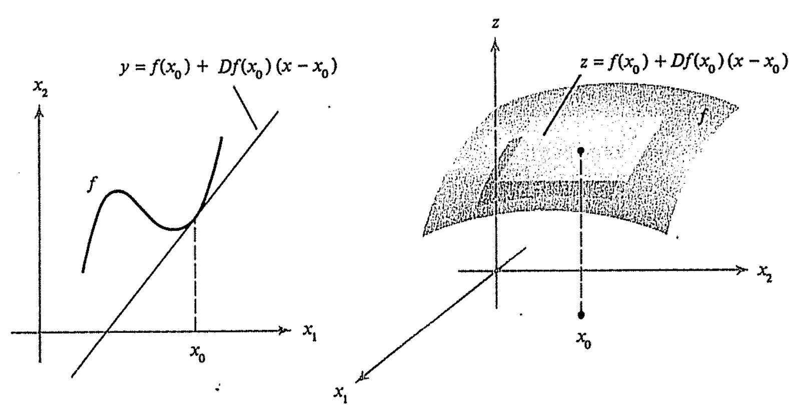
$$Df(x_0) = \left(\frac{\partial f}{\partial x_1}(x_0)\,, \frac{\partial f}{\partial x_2}(x_0) \right) \text{ and the point } x_0 \in A \text{ is given by } x_0 = (x_{01}, x_{02})$$

and $Df(x_0) \cdot (x - x_0)$ is simply the inner-product of two vectors in $\mathbb{R}^2$:

$$Df(x_0) \cdot (x - x_0) = \frac{\partial f}{\partial x_1}(x_{01}, x_{02})\ [x_1 - x_{01}] + \frac{\partial f}{\partial x_2}(x_0)\ [x_2 - x_{02}]\,.$$

(We will discuss this in much greater detail in Section 2.3.)

In Definition 2.2, we are taking a sequence of points $x \in A$ that are approaching the distinguished point $x_0 \in A$. (When we write $x \to x_0$, we really mean that there is a sequence of points $\{x_n\}_{n=1}^\infty$ such that $\lim_{n\to\infty} \|x_n - x_0\| = 0$.)



(a) f: $\mathbb{R} \to \mathbb{R}$ (b) f: $\mathbb{R}^2 \to \mathbb{R}$

We can provide a so-called ϵ-δ definition for the derivative which is equivalent to Definition 2.2.

Definition 2.3 (ϵ-δ). *For every $\epsilon > 0$, there exists $\delta > 0$ such that for $x_0 \in A$, if $\|x-x_0\| < \delta$, then*

$$\|f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)\| \leq \epsilon \|x - x_0\| .$$

(We choose $\delta > 0$ sufficiently small so that $x - x_0 \in A$).

The idea is that with the point $x_0 \in A$ fixed, the function

$$x \mapsto f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)$$

is intended to be the best possible *affine* approximation of the function f at the point $x_0 \in A$. (In this context, an *affine* function is a linear functions plus a constant.)

Definition 2.4 (Differentiability over $A \subset \mathbb{R}^n$). *If f is differentiable at each point $x \in A$, then f is said to be differentiable on A.*

Theorem 2.5. *Let $A \subset \mathbb{R}^n$ be an open set, and suppose that $f : A \to \mathbb{R}^m$ is differentiable at $x_0 \in A$. Then the linear transformation $Df(x_0)$ is uniquely determined by f.*

This theorem asserts that as long as f is differentiable (so that it is not discontinuous), there can be at most one *best linear approximation*. In the case that $f : \mathbb{R}^2 \to \mathbb{R}$, so that the graph of f is a surface in $\mathbb{R}^3$, this means that there is a unique tangent plane to every point on this surface, whenever f is differentiable.

Proof of Theorem 2.5. Let L_1 and L_2 be two linear mappings which both satisfy the conditions of Definition 2.2. We must show that $L_1 = L_2$.

Fix a vector $e \in \mathbb{R}^n$, such that $\|e\| = 1$, and let $x = x_0 + \lambda e$ for $\lambda \in \mathbb{R}$. Geometrically, to get to x from x_0 we move in the direction e a distance λ. Since L_1 and L_2 are both linear, It follows that

$$|\lambda| = \|x - x_0\| \text{ and } \|L_1 \cdot e - L_2 \cdot e\| = \frac{\|L_1 \cdot \lambda e - L_2 \cdot \lambda e\|}{|\lambda|} .$$

Since A is open, the point x is in A whenever λ is taken sufficiently small. By the triangle inequality,

$$\begin{aligned} \|L_1 \cdot e - L_2 \cdot e\| &= \frac{\|L_1 \cdot (x - x_0) - L_2 \cdot (x - x_0)\|}{\|x - x_0\|} \\ &\leq \frac{\|f(x) - f(x_0) - L_1 \cdot (x - x_0)\|}{\|x - x_0\|} + \frac{\|f(x) - f(x_0) - L_2 \cdot (x - x_0)\|}{\|x - x_0\|} . \end{aligned}$$

Notice that by assumption, both terms on the right-hand side of the inequality converge to zero as $\lambda \to 0$, so that $L_1 \cdot e = L_2 \cdot e$. Since e was an arbitrary unit vector (i.e., $\|e\| = 1$), and since for any $y \in \mathbb{R}^n$, $\frac{y}{|y|}$ is also a unit vector, it follows by linearity of L_1 and L_2 that if $L_1 \cdot e = L_2 \cdot$, then $L_1 \cdot y = L_2 \cdot y$ for any vector $y \in \mathbb{R}^n$. Hence, $L_1 = L_2$. $\square$

Example 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. We compute $Df(x)$ and $\frac{df}{dx}$.*

In this case, we know that $\frac{df}{dx}(x) = 3x^2$. Hence, the linear mapping $Df(x)$ is given as

$$h \mapsto Df(x) \cdot h = 3x^2 \cdot h\,.$$

In particular, the linear map consists of multiplication by $3x^2$.

2.2 A reminder of the mean-value theorem for functions of one variable

Theorem 2.7. *If $f : (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$ and f has a maximum (respectively minimum) at c, then $f'(c) = 0$.*

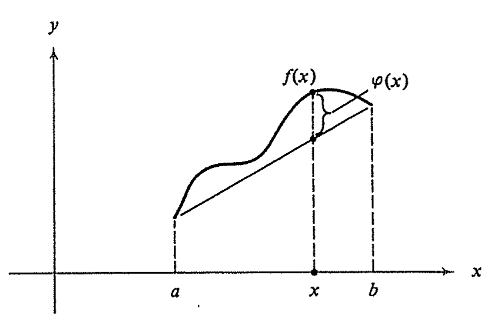
Proof. Let f have a maximum at c. Then for $h \geq 0$, $\frac{f(c+h)-f(c)}{h} \leq 0$, so letting $h \searrow 0$ with $h \geq 0$, we find that $f'(c) \leq 0$. Similarly, with $h \leq 0$, we obtain $f'(c) \geq 0$. Hence $f'(c) = 0$. □

Theorem 2.8 (Rolle's Theorem)**.** *If $f : [a,b]] \to \mathbb{R}$ is continuous, f is differentiable on (a,b) and $f(a) = 0$ and $f(b) = 0$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. If $f(x) = 0$ for all $x \in [a,b]$, then we can choose any c; hence, assume f is not identically zero. Since f is continuous, there is a point c_1 where f achieves its maximum, and a point c_2 where f achieves its minimum. Since $f(a) = 0$ and $f(b) = 0$, at least one of the c_1 or c_2 must lie in the open interval (a,b). If $c_1 \in (a,b)$, we get $f'(c_1) = 0$ by Theorem 2.7, and similarly for c_2. □

Theorem 2.9 (Mean Value Theorem)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b), there is a point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof. Let $\varphi(x) = f(x) - f(a) - (x-a)[f(b) - f(a)]/(b-a)$ and apply Rolle's theorem. □



Corollary 2.10. *If, in addition, $f'(x) = 0$ on (a,b), then f is constant.*

Proof. Applying Theorem 2.9 to f on $[a, x]$, we see that $f(x) - f(a) = f'(c)(x - a) = 0$, so $f(x) = f(a)$ for $x \in [a, b]$, and therefore f is constant. $\square$

This partial list of theorems summarizes sum of the fundamental properties of differentiable functions on the real line. We will generalized these ideas to multiple space dimensions.

Example 2.11. *Let $f : (a, b) \to \mathbb{R}$ be differentiable and $|f'(x)| \le M$. Then $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in (a, b)$. To see this, note that by the mean value theorem,*

$$f(x) - f(y) = f'(c)(x - y)$$

for some $c \in (x, y)$; hence, taking absolute values provides the desired inequality.

2.3 Matrix representation for the derivative $Df(x)$

As we alluded to above, once we choose a basis, we can express the derivative for a function f of multiple variables using components of vectors and matrices to represent linear transformations. With $f : \mathbb{R}^n \to \mathbb{R}^m$ given by (2.2), we can represent the derivative $Df(x_0)$ in terms of its *partial derivatives*. With $f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$, we compute the partial derivatives $\frac{\partial f_j}{\partial x_i}$ for $i = 1, ..., n$ and $j = 1, ..., m$. The notation $\frac{\partial f_j}{\partial x_i}$ means that we compute the usual derivative of f_j with respect to x_i while keeping the other independent variables $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$ fixed.

Definition 2.12 (The partial derivative)**.** *$\frac{\partial f_j}{\partial x_i}(x)$ is given by the following limit, whenever the limit exists:*

$$\frac{\partial f_j}{\partial x_i}(x_1, \ . \ . \ . \ , x_n) = \lim_{h \to 0} \left(\frac{f_j(x_1, ..., x_i + h, ..., x_n) - f_j(x_1, ..., x_i, ..., x_n)}{h} \right) .$$

For each of the m functions f_j with $j = 1, ..., m$, we must compute n partial derivatives.

We have already seen that $Df(x)$ for a function $f : \mathbb{R} \to \mathbb{R}$ is just the linear map consisting of multiplication by df/dx. This fact, which followed from the definition, can be generalized to the following theorem.

Theorem 2.13. *Suppose that $A \subset \mathbb{R}^n$ is an open set, and that $f : A \to \mathbb{R}^m$ is differentiable. Then the partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist, and the matrix of the linear map $Df(x)$ (with respect* *to the standard basis of $\mathbb{R}^n$ and $\mathbb{R}^m$) is given by*

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix} \tag{2.3}$$

where each partial derivative of f is evaluated at $x = (x_1, ..., x_n)$. This matrix of partial derivatives is often called the Jacobian matrix of f.

An important special case is when $m = 1$. f is then a scalar function of n variables, and Df is the gradient of f:

$$Df(x) = \left(\frac{\partial f}{\partial x_1}(x), \ \ldots, \frac{\partial f}{\partial x_n}(x) \right),$$

and the derivative applied to a vector $w = (w_1, ..., w_n)$ is

$$Df(x) \cdot w = \sum_{i=1}^{n} \frac{\partial f}{\partial x_1}(x_1, ..., x_n) w_i .$$

Note well that $Df(x)$ is a linear mapping at each $x \in A$, and that the definition of $Df(x)$ is independent of the basis used. In particular, if we change the standard basis to another one, the entries in the Jacobian matrix would change, but so would the entries in the components of the vector w, and $Df(x) \cdot w$ would be the same.

Another important case occurs when $f : \mathbb{R}^n \to \mathbb{R}^m$ is already a linear map. We denote this by $f = L$. Then from the definition of the derivative, we see $DL = L$, as expected since the best affine approximation of a linear map L is L itself.

It is also of interest to consider the case that f is a constant map. Since each partial derivative must vanish, in this case $Df = 0$.

Example 2.14. *We let $f = \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x_1, x_2) = (f_1, f_2, f_3) = (x_1^2, x_1^3 x_2, x_1^4 x_2^2)$ and compute Df as follows:*

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 & 0 \\ 3x_1^2 x_2 & x_1^3 \\ 4x_1^3 x_2^2 & 2x_1^4 x_2 \end{bmatrix} .$$

Example 2.15. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ denote a linear map (that is, $L(x+y) = L(x) + L(y)$ and $L(\alpha x) = \alpha L(x)$). We show that $DL = L$.*

Given x_0 and $\epsilon > 0$, we must find $\delta > 0$, such that $\|x - x_0\| < \delta$ implies that

$$\|L - L(x_0) - DL(x_0) \cdot (x - x_0)\| \leq \epsilon \|x - x_0\| .$$

But with $DL(x) = L$, the left side of this inequality becomes $\|L - L(x_0) - L(x - x_0)\|$ which is zero by linearity of L. Hence $DL(x) = L$ satisfies the definition for any $\delta > 0$.

Example 2.16. *We compute the gradient of the function $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(x_1, x_2, x_3) = \frac{x_1 \sin x_2}{x_3}$. Since the gradient is $Df = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3})$, where*

$$\frac{\partial f}{\partial x_1} = \frac{\sin x_2}{x_3} , \quad \frac{\partial f}{\partial x_2} = \frac{x_1 \cos x_2}{x_3} , \quad \frac{\partial f}{\partial x_3} = -\frac{x_1 \sin x_2}{x_3^2} .$$

Proof of Theorem 2.13. Our proof is essentially just unwinding definitions from linear algebra about the entries of a matrix. We let e_i denote the standard basis of $\mathbb{R}^n$ and $bbcr^m$.

The jith matrix element of $Df(x)$ is given by the jth component of the vector $Df(x) \cdot e_i$. We denote this jith component by a_{ji}.

Next, let $y = x + he_i$ for some $h \in \mathbb{R}$ and note that

$$\frac{\|f(y) - f(x) - Df(x) \cdot (x - y)\|}{\|y - x\|} = \frac{\|f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_i, ..., x_n) - hDf(x) \cdot e_i\|}{|h|} .$$

By assumption, f is differentiable at x so that the left-hand side goes to zero as $y \to x$, which means that the right-hand side goes to zero as $h \to 0$. Thus, so does the jth component of the numerator, which means that

$$\frac{\|f_j(x_1, ..., x_i + h, ..., x_n) - f_j(x_1, ..., x_i, ..., x_n) - ha_{ji}\|}{|h|} \to 0 \quad \text{as} \quad h \to 0 .$$

Therefore, the partial derivative of f_j with respect to x_i exists and we have that

$$a_{ji} = \lim_{h \to 0} \frac{f_j(x_1, ..., x_i + h, ..., x_n) - f_j(x_1, ..., x_i, ..., x_n)}{h} = \frac{\partial f_j}{\partial x_i} .$$

□

2.4 Continuity of differentiable mappings and differentiable paths

Recall (from lower-division calculus) that differentiable functions on the real line are continuous. The intuition, here, is that having a tangent line (to a curve) at every point is a stronger requirement than not having breaks (discontinuities) in the graph of the function.

For functions $f : (a,b) \to \mathbb{R}$, it is easy to see why differentiability implies continuity. Specifically, if f is differentiable at $x_0 \in (a,b)$, then

$$\begin{aligned}\lim_{x\to x_0} (f(x) - f(x_0)) &= \lim_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0}\right) \cdot (x - x_0) \\ &= f'(x_0) \cdot \lim_{x\to x_0} (x - x_0) = f'(x_0) \cdot 0 = 0\,,\end{aligned}$$

so that $\lim_{x\to x_0}[f(x) - f(x_0)] = 0$, implying that f is continuous at x_0.

There is a natural generalization of this idea to functions $f : \mathbb{R}^n \to \mathbb{R}^m$.

Theorem 2.17. *Suppose $A \subset \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^m$ is differentiable on A. Then f is continuous. In fact, for each $x_0 \in A$, there is a constant $M > 0$ and a $\delta_0 > 0$ such that $\|x - x_0\| < \delta_0$ implies*

$$\|f(x) - f(x_0)\| \leq M\|x - x_0\|\,. \tag{2.4}$$

(A function f that satisfies this inequality is called Lipschitz continuous.)

For the proof, students should recall a fundamental property of linear transformations (matrices) $L : \mathbb{R}^n \to \mathbb{R}^m$, which states that there exists a constant M_0 such that $\|Lx\| \leq M_0\|x\|$ for all $x \in \mathbb{R}^n$. We will employ this inequality for the linear transformation $L = Df(x_0)$. (Note that whenever the linear transformation L has eigenvalues, then M_0 is the largest eigenvalue of L.)

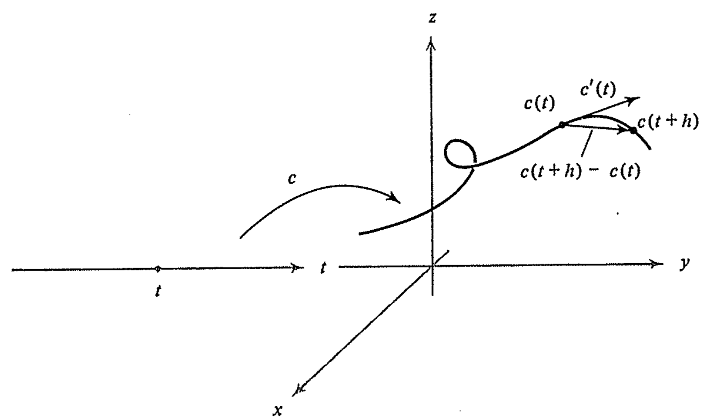
Proof of Theorem 2.17. Continuity of f follows from the Lipschitz condition (2.4), for given $\epsilon > 0$, we can choose $\delta = \min(\delta_0, \frac{\epsilon}{M})$. To do this, let $\epsilon = 1$ in Definition 2.3; then, there exists δ_0 so that $\|x - x_0\| < \delta_0$ implies that

$$\|f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)\| \leq \|x - x_0\|$$

which, in turn, shows (using the triangle inequality) that

$$\|f(x) - f(x_0)\| \leq \|Df(x_0) \cdot (x - x_0)\| + \|x - x_0\|\,.$$

Setting $M = M_0 + 1$, we see that since $\|Df(x_0) \cdot (x - x_0)\| \leq M_0\|x - x_0\|$, the proof is complete. ∎



We have already considered the special case that $f : \mathbb{R}^n \to \mathbb{R}$. Another very important case is that of a parameterized curve $c : \mathbb{R} \to \mathbb{R}^m$, where $c(t)$ represents a curve (or path) in $\mathbb{R}^m$. In this case, the derivative $Dc(t) : \mathbb{R} \to \mathbb{R}^m$ is represented by the vector

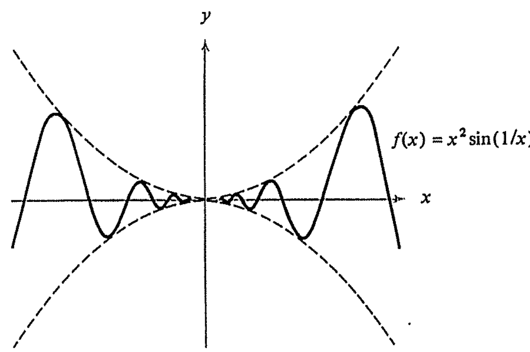
$$\begin{pmatrix} \frac{dc_1}{dt}(t) \\ \cdot \\ \cdot \\ \cdot \\ \frac{dc_m}{dt}(t) \end{pmatrix}$$

where $c(t) = (c_1(t), ..., c_m(t))$. In this case, the vector $Dc(t)$ is usually denoted $c'(t)$ and is called the *tangent vector* (or *velocity vector*) to the curve $c(t)$. We will write the row vector $c'(t)$ (the proper representaion) as the column vector $(c_1'(t), ..., c_m'(t))$ (which is easier to type).

Example 2.18. *With $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$, f is continuous, but f is not differentiable at $x = 0$. To see this, notice that $f(x) = x$ for $x \geq 0$, and $f(x) = -x$ for $x < 0$; hence, f is continuous on $(0, \infty)$ and $(-\infty, 0)$. Since $\lim_{x\to 0} f(x) = 0 = f(0)$, f is also continuous at $x = 0$, and so continuous on all of $\mathbb{R}$. On the other hand, f is not differentiable at $x = 0$, for if it were, then*

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - x_0} = \lim_{x\to 0} \frac{f(x)}{x}$$

would exist. But, for $x > 0$, $f(x)/x = 1$ and for $x < 0$, $f(x)/x = -1$, so the limit cannot exist.



Example 2.19. *The derivative of a function need not be continuous. A simple example is given by*

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0\,. \end{cases}$$

In order to show that f is differentiable at $x = 0$, we must prove that

$$\frac{f(x)}{x} \to 0 \quad as \quad x \to 0\,.$$

Indeed, $|f(x)/x| = |x \sin(1/x)| \leq |x| \to 0$ as $x \to 0$. Thus $f'(0)$ exists and is equal to zero, so that f is differentiable at $x = 0$. However, from basic calculus, we have that

$$f'(x) = 2x \sin(1/x) - \cos(1/x) \quad for \quad x \neq 0\,.$$

As $x \to 0$, the first term on the right-hand side converges to zero, but the second term oscillates between $+1$ and -1, and thus $\lim_{x\to 0} f'(x)$ does not exist. This proves that f" exists but is not continuous at $x = 0$.

Example 2.20. *Let c denote a parameterized curve in $\mathbb{R}^3$, given by $c(t) = (t^2, t, \sin t)$. To find the tangent vector to $c(t)$ at $c(0)$, we compute $c'(t) = (2t, 1, \cos t)$ and evaluate this expression at $t = 0$, to find $c'(0) = (0, 1, 1)$.*

2.5 Criteria for differentiability

From a practical viewpoint, we would like to know if the existence of all of the partial derivatives of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ implies the existence of the derivative Df. Unfortunately, this is not true in general as the following example demonstrates.

Example 2.21. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x_1, x_2) = x_1$ when $x_2 = 0$, $f(x_1, x_2) = x_2$ when $x_1 = 0$, and $f(x_1, x_2) = 1$ elsewhere. Then both $\partial f/\partial x_1$ and $\partial f/\partial x_2$ exist at $x = (0, 0)$*

and are equal to 1*. However, f is not continuous at* $(0,0)$*, so the derivative Df cannot exist at* $(0,0)$*.*

It is not very difficult to understand the failure of the existence of the derivative in the above example. The partial derivatives $\partial f/\partial x_1$ and $\partial f/\partial x_2$ are only checking the convergence of the difference quotients in *two particular directions*: namely, the two axes $\{x_1 = 0\}$ and $\{x_2 = 0\}$, whereas the definition of derivative requires checking convergence of difference quotients in every possible direction (not just lines). Essentially, what is missing in the above example, is continuity of the two partial derivatives that we computed.

Theorem 2.22. *Let $A \subset \mathbb{R}^n$ be an open set and $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, so that $f = (f_1, ..., f_m)$. If each of the partial derivatives $\partial f_j/\partial x_i$ exists and is continuous on A, then f is differentiable on A.*

Proof. By Theorem 2.13, if $Df(x)$ exists, then its matrix representation is given by (2.3). We will show that for $x \in A$ fixed, and for $\epsilon > 0$, there exists $\delta > 0$ such that $\|y - x\| < \delta$ (with $y \in A$) implies that

$$\|f(y) - f(x) + Df(x) \cdot (y - x)\| \le \epsilon \|y - x\| \,. \tag{2.5}$$

Recall that for an m-vector $F \in \mathbb{R}^m$, $\|F\| = \sqrt{\sum_{j=1}^m |F_j|^2}$. Since $\|F\|$ is a sum of m nonnegative terms, it follows that if $\|F\|^2 \le C$ for some constant C, then each component also satisfies $|F_j|^2 \le C$, and conversely, if $|F_j|^2 \le C/m$ then $\|F\|^2 \le C$. Hence, it suffices to prove (2.5) for each component of f, and we choose to start with the first component f_1.

To avoid too many subscripts in our notation, we will denote f_1 by g. Then for $y = (y_1, y_2, ..., y_n)$ and $x = (x_1, x_2, , ..., x_n)$, two points in A, we write

$$\begin{aligned} g(y) - g(x) = g(y_1, y_2, ..., y_n) &- g(x_1, y_2, ..., y_n) + g(x_1, y_2, ..., y_n) - g(x_1, x_2, ..., y_n) \\ &+ g(x_1, x_2, y_3, ..., y_n) - g(x_1, x_2, x_3, ..., y_n) + \cdots \\ &+ g(x_1, x_2, ..., x_{n-1}, y_n) - g(x_1, x_2, ..., x_{n-1}, x_n) \,. \end{aligned}$$

By the mean-value theorem (and with $y_2, y_3, ..., y_n$ fixed), there exists $u_1 \in (x_1, y_1)$ such that

$$\frac{\partial g}{\partial x_1}(u_1, y_2, ..., y_n)(y_1 - x_1) = g(y_1, y_2, ..., y_n) - g(x_1, y_2, ..., y_n) \,,$$

and we can obtain similar expressions for the other n-1 intervals (x_i, y_i), $i = 2, ..., n$. It follows that

$$\begin{aligned} g(y) - g(x) = \frac{\partial g}{\partial x_1}(u_1, y_2, ..., y_n)(y_1 - x_1) &+ \frac{\partial g}{\partial x_2}(y_1, u_2, ..., y_n)(y_2 - x_2) \\ + \cdots + \frac{\partial g}{\partial x_n}(y_1, y_2, ..., u_n)(y_n - x_n) \,. \end{aligned}$$

Since

$$Dg(x)\cdot(y-x)=\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(x_1,x_2,...,x_n)(y_i-x_i)\,,$$

the triangle inequality shows that

$$\begin{aligned}&\|g(y)-g(x)-Dg(x)\cdot(y-x)\|\\&\quad\le\left\{\left|\frac{\partial g}{\partial x_1}(u_1,y_2,...,y_n)-\frac{\partial g}{\partial x_1}(x_1,x_2,...,x_n)\right|\right.\\&\qquad\left.+\cdots+\left|\frac{\partial g}{\partial x_1}(x_1,...,x_{n-1},y_n)-\frac{\partial g}{\partial x_1}(x_1,...,x_{n-1},x_n)\right|\right\}\ \|y-x\|\,. \qquad (2.6)\end{aligned}$$

Since $\partial g/\partial x_1=\partial f_1/\partial x_1$ is assumed to be continuous, if $\|y-x\|<\delta$ for $\delta>0$ taken sufficiently small, we have that

$$\left|\frac{\partial g}{\partial x_i}(y_1,y_2,...,y_n)-\frac{\partial g}{\partial x_i}(x_1,x_2,...,x_n)\right|\le\frac{\epsilon}{m^{\frac{3}{2}}}\quad\text{for each}\quad i=1,2,...,n\,.$$

Since each u_i lies between x_i and y_i, we see that the terms in braces in (2.6) are bounded by $\epsilon/\sqrt{m}$ and hence we see that

$$\|f(y)-f(x)-Df(x)\cdot(y-x)\|\le\epsilon\|y-x\|$$

as desired. □

2.6 The directional derivative

In multiple space dimensions, it is of great interest to be able to measure the rate-of-change of functions in any given direction.

Definition 2.23 (Directional Derivative)**.** *Let U denote an open neighborhood of $x_0\in\mathbb{R}^n$, and suppose that $f:U\to\mathbb{R}$. Let $e\in\mathbb{R}^n$ denote a unit vector (so that $\|e\|=1$). Then*

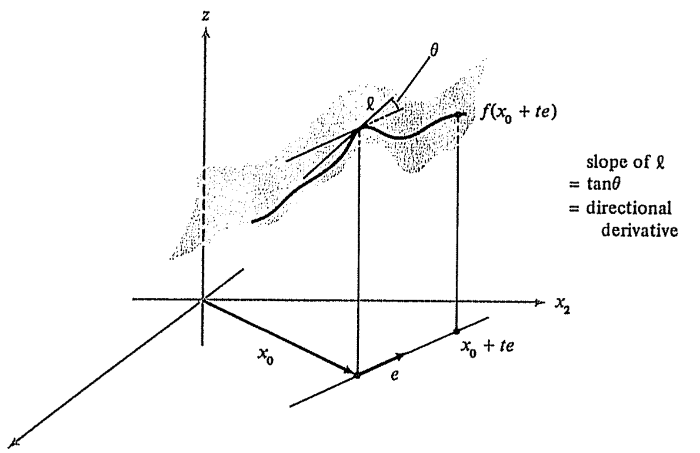
$$\left.\frac{d}{dt}f(x_0+te)\right|_{t=0}=\lim_{t\to0}\frac{f(x_0+te)-f(x_0)}{t}$$

is called the directional derivative of f at x_0 in the direction e.

From the definition and the figure, we see that the directional derivative is indeed the rate of change of f in the direction e.

In particular, we claim that the directional derivative of f at x_0 in the direction e is, in fact, $Df(x_0)\cdot e$. To see this, we need only examine the definition of derivative with the nearby point x taken to be $x=x_0+te$. We find that

$$\left\|\frac{f(x_0+te)-f(x_0)}{t}-Df(x_0)\cdot e\right\|\le\epsilon\|e\|\quad\text{for any }\epsilon>0$$



if $|t|$ is sufficiently small. Thus, whenever f is differentiable at x_0, then the *directional derivatives* also exist and are given by

$$\lim_{t\to 0}\frac{f(x_0+te)-f(x_0)}{t}=Df(x_0)\cdot e\,.$$

Specific examples of the directional derivative are the partial derivatives: for each $i = 1,...,n$, $\partial f/\partial x_i$ is the derivative of f in the direction of the ith coordinate axis $e_i = (0,0,..., \underbrace{1}_{i\text{th slot}}, ...,0,0)$.

Notice that for a function $f : \mathbb{R}^2 \to \mathbb{R}$, the directional derivative $Df(x_0)\cdot e$ can be used to determine the plane tangent to the graph of f. Namely, the line l, $z = f(x_0)+Df(x_0)\cdot te$, is the tangent to the graph of f since as in the figure, $Df(x_0)\cdot e$ is just the rate of change of f in the direction e. Thus, the tangent plane to the graph of f at $(x_0, f(x_0))$ is given by the equation

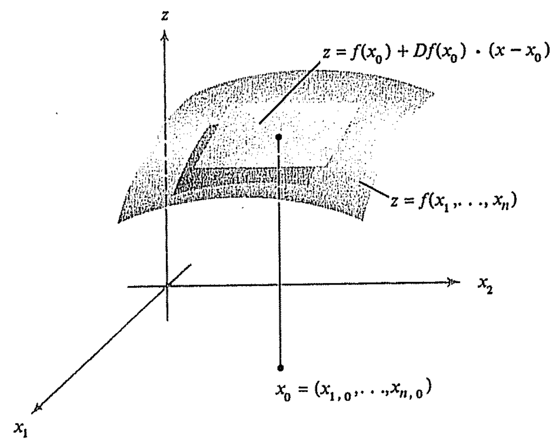
$$z = f(x_0) + Df(x_0)\cdot(x - x_0)\,.$$

This equation can be used to provide a rigorous definition of the tangent plane to a surface.

Again, it may be tempting to believe that the existence of all directional derivatives at a point might imply differentiability at that point, but this is not the case.

Example 2.24. *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is given by*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y}, & x^2 \neq -y\,,\\ 0, & x^2 = -y\,.\end{cases}$$



Then if $e = (e_1, e_2)$ and $e_2 \neq 0$, then

$$\frac{1}{t} f(te_1, te_2) = \frac{1}{t} \frac{t^2 e_1 e_2}{t^2 e_1^2 + te_2} = \frac{e_1 e_2}{te_1^2 + e_2} \to e_1 \quad as \quad t \to 0\,,$$

while if $e_2 = 0$, then $\frac{1}{t} f(te_1, te_2) \to 0$ as $t \to 0$. Hence, each directional derivative exists at $(0,0)$, but f is not continuous at $(0,0)$ since for x^2 near $-y$ with both x, y small, f is very large.

Given $\delta > 0$ and M, choose (x, y) such that $x^2 = -y + \epsilon$ and $\|(x, y)\| < \delta$. Then $f(x, y) = xy/\epsilon$, which for $\epsilon > 0$ taken small, can be made larger than M. Thus f is not bounded on the ball of radius δ, $\{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} < \delta\}$, for any $\delta > 0$. It follows that f is not continuous at $(0,0)$, so by Theorem 2.17, f is not differentiable at $(0,0)$.

This example shows that the existence of all directional derivatives of f does not even imply continuity of f, let alone differentiability.

Example 2.25. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + y$. Let us compute the tangent plane to the graph(f) at $x = 1$ and $y = 2$. Since*

$$Df(x, y) = (\partial f/\partial x, \partial f/\partial y) = (2x, 1)\,,$$

we see that $Df(1, 2) = (2, 1)$. Thus, the equation for the tangent plane at the point $(1, 2)$ becomes

$$z = 3 + (2, 1) \begin{pmatrix} x - 1 \\ y - 2 \end{pmatrix} = 3 + 2(x - 1) + (y - 2)\,,$$

or

$$2x + y - z = 1\,.$$

2.7 The chain-rule and product-rule

2.7.1 Chain-rule

The chain-rule is one of the most important tools in analysis because it provides an algorithm for differentiating composite functions which most commonly arise from changing (or mapping) the independent variables and the domains over which they are defined. Some of the most important formulas, identities, and solutions to differential equaions are obtained by a very carefully (and clever) change of variables, and the chain-rule shows how these various equations can be transformed.

Students may recall from vector calculus, that if $u(x, y)$ and $v(x, y)$ are real-valued differentiable functions of two variables, and if $f(u, v)$ is also a real-valued differentiable functions of two variables, then

$$\frac{\partial}{\partial x} f(u(x,y), v(x,y)) = \frac{\partial f}{\partial u}(u(x,y), v(x,y))\frac{\partial u}{\partial x}(x,y) + \frac{\partial f}{\partial v}(u(x,y), v(x,y))\frac{\partial v}{\partial x}(x,y)\,.$$

In order to generalize this, we define the composition of two functions.

Definition 2.26 (Composite function)**.** *Let $A \subset \mathbb{R}^n$ denote an open set, and let $B \subset \mathbb{R}^m$ denote an open set. Suppose that $f : A \to \mathbb{R}^m$, $g : B \to \mathbb{R}^p$, and that $f(A) \subset B$. Then the composite function $g \circ f : A \to \mathbb{R}^p$ is defined as*

$$[g \circ f](x) = g(f(x))\,.$$

Theorem 2.27 (Chain-rule)**.** *Let $f : A \to \mathbb{R}^m$ be differentiable on the open set $A \subset \mathbb{R}^n$ and $g : B \to \mathbb{R}^p$ be differentiable on the open set $B \subset \mathbb{R}^m$, and suppose that $f(A) \subset B$. Then the composite function $g \circ f : A \to \mathbb{R}^p$ is differentiable on A and for $x_0 \in A$,*

$$D[f \circ g](x_0) = Dg(f(x_0) \cdot Df(x_0)\,.$$

Notice that by definition, $Df(x_0) : \mathbb{R}^n \to \mathbb{R}^m$ and $Dg(f(x_0)) : \mathbb{R}^m \to \mathbb{R}^p$ so that $Dg(f(x_0)) \cdot Df(x_0) : \mathbb{R}^n \to \mathbb{R}^p$ is well-defined. With respect to the standard basis of vectors e_i (for Euclidean space), we can express $Dg(f(x_0)) \cdot Df(x_0)$ as the multiplication of two matrices. If $h = g \circ f$, then

$$Dh(x) = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial g_p}{\partial y_1} & \cdots & \frac{\partial g_p}{\partial y_m} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

where $\partial g_i/\partial y_j$ are evaluated at $y = f(x)$, and $\partial f_j/\partial x_k$ are evaluated at x. Thus, in order to compute the kth partial derivative of the ith component of the vector h, we write

$$\frac{\partial h_i}{\partial x_k}(x) = \sum_{j=1}^{m} \frac{\partial g_i}{\partial y_j}(f(x))\, \frac{\partial f_j}{\partial x_k}(x) \quad i = 1, ..., p, \quad k = 1, ..., m\,.$$

Example 2.28 (Polar coordinates)**.** *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$, so that $f(x, y)$ is real-valued, and let $x(r, \theta) = r\cos\theta$ and $y(r, \theta) r \sin\theta$. The pair (r, θ) are called polar coordinates. If we apply the chain rule to the composite function*

$$f(x(r, \theta), y(r, \theta))\,,$$

we find

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\,,$$

and

$$\frac{\partial f}{\partial \theta} = -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta\,.$$

Hence, we see that the chain-rule allows us to derive the famous identities

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\,.$$

Example 2.29. *Suppose that $f(u, v, w) = u^2 v + w v^2$, $u(x, y) = xy$, $v(x, y) = \sin x$, and $w(x, y) = e^x$. If we form the composite function $h(x, y) = f(u(x, y), v(x, y), w(x, y))$ given by*

$$h(x, y) = x^2 y^2 \sin x + e^x \sin^2 x\,,$$

then we can directly compute the partial derivative

$$\frac{\partial h}{\partial x} = 2xy^2 \sin x + x^2 y^2 \cos x + e^x \sin^2 x + 2e^x \sin x \cos x\,.$$

On the other hand, by the chain-rule, we can also compute that

$$\begin{aligned}
\frac{\partial h}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} \\
&= 2uv\frac{\partial u}{\partial x} + (u^2 + 2vw)\frac{\partial v}{\partial x} + v^2\frac{\partial w}{\partial x} \\
&= xy^2 \sin x + x^2 y^2 \cos x + e^x \sin^2 x + 2e^x \sin x \cos x\,.
\end{aligned}$$

Example 2.30. *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $F : \mathbb{R}^2 \to \mathbb{R}$ also be differentiable. We set*

$$F(x, y) = f(xy)\,.$$

By the chain-rule, $\frac{\partial F}{\partial x} = f'(xy)y$ and $\frac{\partial F}{\partial y} = f'(xy)x$, which shows that $x\frac{\partial F}{\partial x} = y\frac{\partial F}{\partial y}$.

Proof of Theorem 2.27(The Chain-Rule). To show that $D(g\circ f)(x_0) = Dg(f(x_0))\cdot Df(x_0)$, we must prove that

$$\lim_{x\to x_0} \frac{\|g\circ f(x) - g\circ f(x_0) - [Dg(f(x_0))\cdot Df(x_0)]\cdot(x-x_0)\|}{\|x-x_0\|} = 0\,.$$

We proceed by estimating the numerator as follows:

$$\begin{aligned}
&\|g\circ f(x) - g\circ f(x_0) - Dg(f(x_0))\cdot Df(x_0)\cdot(x-x_0)\| \\
&\qquad = \|g(f(x) - g(f(x_0)) - Dg(f(x_0))\cdot(f(x)-f(x_0)) \\
&\qquad\qquad + Dg(f(x_0))\cdot(f(x)-f(x_0)) - Df(x_0)\cdot(x-x_0)\| \\
&\qquad \le \|g(f(x) - g(f(x_0)) - Dg(f(x_0))\cdot(f(x)-f(x_0))\| \\
&\qquad\qquad + \|Dg(f(x_0))\cdot[f(x)-f(x_0) - Df(x_0)\cdot(x-x_0)]\|
\end{aligned}$$

by the triangle inequalty. Since f is differentiable, by Theorem 2.17 there exists $\delta_0 > 0$ and $M > 0$ such that $\|f(x) - f(x_0)\| \le M\|x-x_0\|$ whenever $\|x-x_0\| < \delta_0$. By the definition of the derivative, for $\epsilon > 0$ given, there exists $\delta_1 > 0$ such that $\|y - f(x_0)\| < \delta_1$ implies that

$$\|g(y) - g(f(x_0)) - Dg(f(x_0))\cdot(y - f(x_0))\| \le \left(\frac{\epsilon}{2M}\right)\|y - f(x_0)\|\,.$$

Next, we set $\delta_2 = \min(\delta_0, \delta_1)$. Then, for $\|x - x_0\| < \delta_2$, we have that

$$\frac{\|g(f(x)) - g(f(x_0)) - Dg(f(x_0))\cdot(f(x)-f(x_0))\|}{\|x-x\|} < \frac{\epsilon}{2}\,.$$

Since $Dg(f(x_0))$ is a linear map, we know that there is a constant $\tilde{M} > 0$ such that $\|Dg(f(x_0))\cdot y\| \le \tilde{M}\|y\|$ for all $y\in\mathbb{R}^m$. By definition of the derivative, there exists $\delta_3 > 0$ such that $\|x - x_0\| < \delta_3$ implies that

$$\frac{\|f(x) - f(x_0) - Df(x_0)\cdot(x-x_0)\|}{\|x-x_0\|} < \frac{\epsilon}{2\tilde{M}}\,.$$

Then $\|x-x_0\| < \delta_3$ implies that

$$\begin{aligned}
&\frac{\|Dg(f(x_0))\cdot[f(x) - f(x_0) - Df(x_0)\cdot(x-x_0)]\|}{\|x-x_0\|} \\
&\qquad\qquad \le \frac{\tilde{M}\ \|f(x) - f(x_0) - Df(x_0)\cdot(x-x_0)]\|}{\|x-x_0\|} < \frac{\epsilon}{2}\,.
\end{aligned}$$

Finally, choosing $\delta = \min(\delta_2, \delta_3)$, we see that $\|x - x_0\| < \delta$ implies that

$$\begin{aligned}
&\frac{\|g \circ f(x) - g \circ f(x_0) - Dg(f(x_0)) \cdot Df(x_0) \cdot (x - x_0)\|}{\|x - x_0\|} \\
&\qquad \le \frac{\|g(f(x)) - g(f(x_0)) - Dg(f(x_0)) \cdot (f(x) - f(x_0))\|}{\|x - x\|} \\
&\qquad\quad + \frac{\|Dg(f(x_0)) \cdot [f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)]\|}{\|x - x_0\|} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.
\end{aligned}$$

□

2.7.2 Product-rule

At almost the same time that Sir Isaac Newton had developed calculus in England, a German mathematician named Gottfried Leibniz had independently developed the fundamentals of calculus, and the product-rule of differentiation is commonly termed the *Liebniz rule.*

Theorem 2.31 (Product-rule)**.** *Let $A \subset \mathbb{R}^n$ be open and let $f : A \to \mathbb{R}^m$ and $g : A \to \mathbb{R}$ be differentiable functions. Then the product gf is also differentiable, and for $x \in A$ the derivative $D(fg)(x)$ exists and is a linear transformation of $\mathbb{R}^n$ to $\mathbb{R}^m$, given by*

$$D(gf)(x) \cdot w = g(x)\,[Df(x) \cdot w] + [Dg(x) \cdot w]\, f(x) \quad \text{for all} \quad w \in \mathbb{R}^n\,. \tag{2.7}$$

Remark 2.32. *Notice that while f is vector-valued (taking values in $\mathbb{R}^m$), the function g is scalar-valued (taking values in $\mathbb{R}$). We understand what it means to multiply a real number with a vector: the real number multiplies each component of the vector. On the other hand, multiplication of two vectors can have many meanings, with the scalar dot product and the vector cross product being just two examples.*

Remark 2.33. *It is common to abbreviate the identity (2.7) as*

$$D(gf) = g\ Df + Dg\ f\,,$$

removing the explicit dependence on the independent variables, but the meaning is always as given in (2.7).

Students may recall the product-rule from lower-division calculus. Expressed in terms of the components of the vectors x and $f(x)$, the product-rule can be expressed as

$$\frac{\partial}{\partial x_i}(g\, f_j) = g \frac{\partial f_j}{\partial x_i} + \frac{\partial g}{\partial x_i} f_j\,,$$

so that for each $i = 1, ..., n$ and $j = 1, ..., m$ fixed, we have the same identity as in the one-variable case.

The other differentiation rules are encapsulated by the statement that D is linear, meaning that

1. $D(u+v) = Du + Dv$ for all $u : \mathbb{R}^n \to \mathbb{R}^m$ and $v : \mathbb{R}^n \to \mathbb{R}^m$ that are differentiable;
2. $D(\alpha u) = \alpha Du$ for all $\alpha \in \mathbb{R}$ and $u : \mathbb{R}^n \to \mathbb{R}^m$ differentiable.

Proof of the Theorem 2.31. Our objective is to show that the product gf is differentiable at say $x_0 \in A$, and that $g(x_0)Df(x_0) \cdot (x - x_0) - [Dg(x_0) \cdot (x - x_0)]\, f(x_0)$ satisfies the definition of the derivative of fg, hence showing that (2.7) holds (with w given by $x - x_0$). In anticipation of the use of the triangle inequality, we first write some needed inequalities.

Since f and g are both differentiable, given $\epsilon > 0$ and $x_0 \in A$, we choose $\delta > 0$ sufficiently small so that if $\|x - x_0\| < \delta$, then

(i) $|g(x)| \le |g(x_0)| + 1 = M$;

(ii) $\|f(x) - f(x_0) - Df(x_0) \cdot (x - x_0)\| \le \frac{\epsilon}{3M} \|x - x_0\|$;

(iii) $\|g(x) - g(x_0) - Dg(x_0) \cdot (x - x_0)\| \le \frac{\epsilon}{3\|f(x_0)\|} \|x - x_0\|$;

(iv) $\|g(x) - g(x_0)\| \le \frac{\epsilon}{3M}$.

Recall, that from linear algebra, $\|Df(x_0) \cdot w\| \le M\|w\|$ for all $w \in \mathbb{R}^n$. (The inequalities (iii) and (iv) are needed only if $f(x_0) \ne 0$ and $Df(x_0) \ne 0$.)

Using the triangle inequality, we see that for $\|x - x_0\| < \delta$,

$$
\begin{aligned}
&\|g(x)f(x) - g(x_0)f(x_0) - g(x_0)Df(x_0) \cdot (x - x_0) - [Dg(x_0) \cdot (x - x_0)]\, f(x_0)\| \\
&\quad \le \|g(x)f(x) - g(x)f(x_0) - g(x)Df(x_0) \cdot (x - x_0)\| \\
&\qquad + \|g(x)Df(x_0) \cdot (x - x_0) - g(x_0)Df(x_0) \cdot (x - x_0)\| \\
&\qquad + \|g(x)f(x_0) - g(x_0)f(x_0) - [Dg(x_0) \cdot (x - x_0)]f(x_0)\| \\
&\quad \le M\frac{\epsilon\|x - x_0\|}{3M} + \frac{\epsilon}{3M}M\|x - x_0\| + \frac{\epsilon\|x - x_0\|}{3\|f(x_0)\|}\|f(x_0)\| \\
&\quad = \epsilon\|x - x_0\| \,.
\end{aligned}
$$

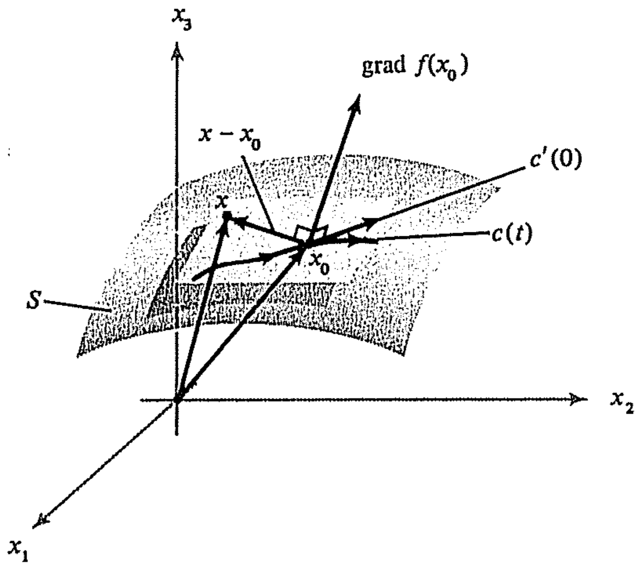
□

2.8 The geometry of the gradient

Let $A \subset \mathbb{R}^n$ denote an open set, and let $: A \to \mathbb{R}$ be a differentiable function. With respect to the standard basis of $\mathbb{R}^n$, the gradient of f can be written as

$$
\operatorname{grad} f(x) = Df(x) = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right) .
$$

Hence, the directional derivative of f at x_0 in the direction e can be written in terms of



the gradient of f as follows:

$$\left.\frac{d}{dt}\right|_{t=0} f(x_0 + te) = \operatorname{grad} f(x_0) \cdot e\,.$$

In order to understand the geometry of the gradient of f, we consider the surface S given by

$$S = \{x \in \mathbb{R}^n \mid f(x) = c = \text{constant}\}\,.$$

For example suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ and $f(x) = \|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. In this case for every constant $a > 0$,

$$S = \{x \in \mathbb{R} \mid \|x\| = a\} \text{ denotes the sphere of radius } a.$$

Each such surface S is called the level set of f, and denotes the set of points $x \in \mathbb{R}^n$ on which the function takes a constant value a.

We claim that if $x \in S$, then $\operatorname{grad} f(x)$ is a vector that is *orthogonal* to the surface S at the point x. To see this, let $c(t)$ denote a continuously differentiable curve lying on the surface S. Since the curve$c(t)$ is differentiable, there is a tangent vector $dc/dt(t)$ at every point t of the curve, which we denote by $c'(t)$. Next, suppose that the point $x_0 = c(0)$; then since $c'(0)$ is a vector tangent to the curve, and since the curve is lying on the surface S, the vector $c'(0)$ is also tangent to the surface S at the point x_0. With this terminology in place, our claim can be restated as

$$\operatorname{grad} f(x_0) \cdot c'(0) = 0\,.$$

The proof is straightforward. We consider the composite function $f(c(t))$ which is merely the function evaluated along the curve $c(t)$. Since $c(t)$ lies on the level surface S, along which f takes the constant value a, we see that

$$f(c(t)) = a\,.$$

Differentiating this identity with respect to t, the chain-rule tells us that

$$Df(c(t)) \cdot c'(t) = 0 \quad \text{or equivalently} \quad \operatorname{grad} f(c(t)) \cdot c'(t) = 0\,.$$

Evaluating this expression at $t = 0$, and using the fact that $c(0) = x_0$, we find that $\operatorname{grad} f(x_0) \cdot c'(0) = 0$, which proves the claim.

Note, that we may describe the tangent plane to S at x_0 by

$$\operatorname{grad} f(x_0) \cdot (x - x_0) = 0\,, \tag{2.8}$$

since $\operatorname{grad} f(x_0)$ is orthogonal to S. We can write this

We also have, from the definition of the inner-product, that

$$\operatorname{grad} f(x_0) \cdot e = \|\operatorname{grad} f(x_0)\| \cos\theta\,,$$

where $\|e\| = 1$ and θ is the angle between $\operatorname{grad} f(x_0)$ and the unit-vector e. It is evident that $\operatorname{grad} f(x_0)$ is the *direction* along which the function f is changing the fastest. It is often stated that moving along the direction of $\operatorname{grad} f$ is *the path of steepest descent.*

Example 2.34. *Since* $\operatorname{grad} f$ *is a vector which is orthogonal to the level set surface of a function* f*, we can compute the unit normal vector to such a surface. Suppose we consider the sphere of radius* 3 *in* $\mathbb{R}^3$ *given by*

$$f(x, y, z) = x^2 + y^2 + z^2 = 3\,,$$

and determine the unit normal at the point $x_0 = (1, 1, 1)$ *on this surface.*

Since $\operatorname{grad} f = (2x, 2y, 2z)$*, it follows that* $\operatorname{grad} f(x_0) = (2, 2, 2)$*. This vector is pointing in the normal direction, but it is not normalized to have unit length. The unit normal is*

$$N(x_0) = \frac{\operatorname{grad} f(x_0)}{\|\operatorname{grad} f(x_0)\|}\,,$$

so that $N(x_0) = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$*.*

Example 2.35. *We can also compute the tangent plane to a level surface of* f*. Suppose that the surface* S *is given by* $f(x_1, x_2, x_3) = x_1^2 - x_2^2 + x_1 x_3 = 2$*, and we would like to compute the tangent plane to* S *at the point* $x_0 = (1, 0, 1)$*. Since* $\operatorname{grad} f(x_0) = (3, 0, 1)$*, we see that the identity (2.8) shows that the tangent plane is given by*

$$0 = \operatorname{grad} f(x_0) \cdot (x - x_0) = 3(x_1 - 1) + 1(x_3 - 1) \quad \textit{or} \quad 3x_1 + x_3 = 4\,.$$

2.9 The Mean Value Theorem

In Theorem 2.9, we stated the mean value theorem for functions $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$: if $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b)$ is differentiable, then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Unfortunately, for $f : \mathbb{R}^n \to \mathbb{R}^m$, this version of the mean-value theorem does not hold. As we discussed in lecture, if $f : \mathbb{R} \to \mathbb{R}^2$ is given by $f(x) = (x^2, x^3)$, then it is not possible to find a point $c \in (0, 1)$ such that $f(1) - f(0) = Df(c)(1 - 0)$, for this would mean that $(1, 1) - (0, 0) = (2c, 3c^2)$ and c would have to satisfy both $2c = 1$ and $3c^2 = 1$ which is not possible.

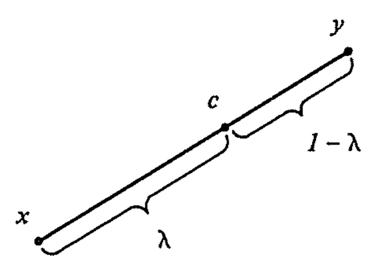
On the other hand, our intuition suggests that if we consider the rate of change of a function in a given direction, the mean value theorem should somehow still hold if we consider the directional derivative in the direction of the line joining two points x and y in $\mathbb{R}^n$. In order to proceed, we first

$$\text{assume that } f \text{ is real-valued, so that } f : \mathbb{R}^n \to \mathbb{R}\,.$$

For intervals in (a, b) in $\mathbb{R}$, we take for granted what it means for a point c to be "between" the points x and y. In higher dimensions, we must give a precise definition of this.

Definition 2.36 (c is "between" x and y for $c, x, y \in \mathbb{R}^n$). *We say that c is on the line segment joining x and y, or is between x and y if*

$$c = (1 - \lambda)x + \lambda y \quad \textit{for some} \quad 0 \le \lambda \le 1\,.$$



Theorem 2.37 (Mean Value Theorem for $f : \mathbb{R}^n \to \mathbb{R}$). *Suppose that $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable on an open set A. For any $x, y \in A$ such that the line segment joining x and y lies in A (which need not happen for any two points x, y), there is a point c on that segment such*

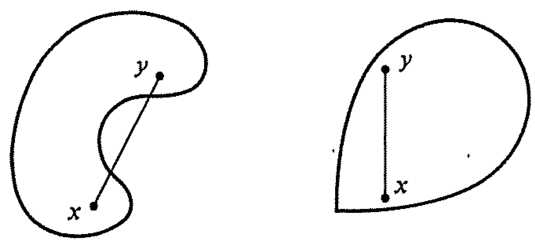
$$f(y) - f(x) = Df(c) \cdot (y - x)\,.$$

Next, for vector-valued functions $f : \mathbb{R}^n \to \mathbb{R}^m$, we can generalize Theorem 2.37 to the following:

Theorem 2.38 (Mean Value Theorem for $f : \mathbb{R}^n \to \mathbb{R}^m$). *Suppose that $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on an open set A. Suppose that the line segment joining x and y lies in A and $f = (f_1, ..., f_m)$. Then there exists a vector $c = (c_1, ..., c_m)$ on that segment such*

$$f_j(y) - f_j(x) = Df_j(c_j) \cdot (y - x) \qquad j = 1, ..., m\,.$$

Definition 2.39 (Convex sets). *A set $A \subset \mathbb{R}^n$ is said to be convex if for each $x, y \in A$, the segment joining x, y also lies in A.*



not convex (a) convex (b)

Example 2.40. *Let $A \subset \mathbb{R}^n$ be an open convex set and let $f : A \to \mathbb{R}^m$ be differentiable. If $Df = 0$, then we can prove that f is constant.*

For $x, y \in A$, and for each component f_j of the vector-valued function $f = (f_1, ..., f_m)$, there is a vector c_j such that

$$f_j(y) - f_j(x) = Df_j(c_j) \cdot (y - x)\,.$$

Since $Df = 0$, it follows that $Df_j = 0$ for each component f_j, and so $f_j(y) = f_j(x)$. It follows that $f(y) = f(x)$, so that f is constant.

Example 2.41. *Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous, $f(0) = 0$, $f : (0, \infty)$ is differentiable, and f' is non-decreasing. We prove that the function $g(x)$ given by $g(x) = \frac{f(x)}{x}$ is non-decreasing on $(0, \infty)$ as well.*

According to the mean value theorem, a function $h : \mathbb{R} \to \mathbb{R}$ is non-decreasing if $h'(x) \geq 0$, because $x \leq y$ implies that

$$h(y) - h(x) = h'(c)(y - x) \geq 0\,.$$

Now

$$g'(x) = \frac{xf'(x) - f(x)}{x^2}$$

and

$$f(x) = f(x) - f(0) = f'(c) \cdot x \le xf'(x)$$

since $0 < c < x$ *and* $f'(c) \le f'(x)$. *Thus* $xf'(x) - f(x) \ge 0$, *so* $g'(x) \ge 0$, *which implies that* g *is non-decreasing.*

Proof of Mean Value Theorem 2.37. For $0 \le t \le 1$, we consider the segment $(1-t)x + ty$ joining the points x and y. Notice that when $t = 0$ we recover x, while when $t = 1$ we recover y, so the parameter t moves us in a straight-line from point x to point y as t increases from 0 to 1.

Consider the function $h : [0,1] \to \mathbb{R}$ defined by

$$h(t) = f((1-t)x + ty)\,.$$

By the chain-rule and the assumed differentiability of f on the set A, the function h is differentiable in t on the open interval $(0,1)$. By the mean value theorem for real-valued functions of one space variable (Theorem 2.9), there exists a point $t_0 \in (0,1)$ such that $h(1) - h(0) = h'(t_0)(1-0)$. On the other hand, $h(1) = f(y)$ and $h(0) = f(x)$. We use the chain-rule to compute $h'(t_0)$. We see that

$$h'(t_0) = Df((1-t)x + ty) \cdot (y - x)\,,$$

since

$$\frac{d}{dt}[(1-t)x + ty] = y - x\,.$$

It follows that the desired point c in the statement of theorem can be chosen as

$$c = (1-t_0)x + t_0 y\,.$$

$\square$

Proof of Mean Value Theorem 2.38. The proof follows by applying the proof of Theorem 2.37 to each component of f separately. $\square$

2.10 Higher-order derivatives and C^k-class functions

We begin with a discussion of higher-order derivatives for functions $f : \mathbb{R}^n \to \mathbb{R}^m$. This is accomplished by iterating the partial derivative; for example, if $f : \mathbb{R}^2 \to \mathbb{R}$, then

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2} f\right)\,.$$

More generally, the second derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ is obtained (when it exists) by differentiating Df.

Definition 2.42. *Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denote the space of linear transformations (or maps) from $\mathbb{R}^n$ to $\mathbb{R}^m$. (Given a basis for $\mathbb{R}^n$ and $\mathbb{R}^m$, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ can be identified with the $m \times n$ matrices, and hence with $\mathbb{R}^{nm}$.)*

Recall, that if $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable on the open set A, then

$$Df : A \subset \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)\,, \tag{2.9}$$

which is merely a restatement of the fact that for each $x_0 \in A$, $Df(x_0) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. On the other hand, (2.9) allows us to clearly understand the second derivative. If we differentiate (2.9) at the point $x_0 \in A$, the definition of derivative shows that $D(Df)(x_0)$ is a linear map from $\mathbb{R}^n$ to $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. We write

$$D(Df)(x_0) = D^2 f(x_0).$$

Now, since $D^2 f(x_0) : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, then for each vector $u \in \mathbb{R}^n$, $D^2 f(x_0) \cdot u \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. This means that if we take any other vector $v \in \mathbb{R}^n$, then

$$(D^2 f(x_0) \cdot u) \cdot v \in \mathbb{R}^m\,.$$

It is convenient to identify $D^2 f(x_0)$ with a bilinear map which maps $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ and write

$$(D^2 f(x_0) \cdot u) \cdot v = D^2 f(x_0)(u, v) \quad \forall \quad u, v \in \mathbb{R}^n\,.$$

Recall that a *bilinear map* $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is a map which is linear in each of its variables separately, so that for $u, v \in \mathbb{R}^n$, the map $u \mapsto B(u, v)$ is linear for v fixed, and the map $v \mapsto B(u, v)$ is linear for u fixed.

In the special case that the bilinear map takes values is real-valued, so that $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, we can associate to each bilinear map an $n \times n$ matrix (in a basis for $\mathbb{R}^n$). In particular, letting e_i denote the standard basis of $\mathbb{R}^n$, the ijth component of the matrix representing B is given by

$$B_{ij} = B(e_i, e_j)\,.$$

Then, if we write $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$, we see that

$$\begin{aligned} B(u, v) &= \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} u_i v_j \\ &= (u_1, ..., u_n) \begin{pmatrix} B_{11} & \cdots & B_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ B_{n1} & \cdots & B_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix}. \end{aligned}$$

When $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable at x_0, we can write $D^2 f(x_0)$ as an $n \times n$ matrix in the standard basis of $\mathbb{R}^n$.

Theorem 2.43 ($D^2f(x_0)$ in matrix form)**.** *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be twice differentiable on the open set A. Then the matrix of $D^2f(x_0)$ ($x_0 \in A$) is given by*

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

where each second-order partial derivative is evaluated at $x_0 \in A$.

Proof. Since $f : A \to \mathbb{R}$, $Df(x)$ is given in the standard basis as $(\partial f/\partial x_1, ..., \partial f/\partial x_n)$. Thus, we simply apply Theorem 2.13 to Df to conclude the proof. □

Each component of this matrix is written as $\partial^2 f/\partial x_i \partial x_j$ where the indices i, j can range from 1 to n.

For higher-order derivatives, we proceed in an similar manner. For example, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is three-times differentiable, then $D^3f(x_0)$ is a trilinear map: for each x_0,

$$D^3f(x_0) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \,.$$

A trilinear map is a map which is linear in each of its variables separately. We are not able to associate a matrix with $D^3f(x_0)$, but we can express it in component form as $\partial^3 f/\partial x_i \partial x_j \partial x_k$ where the indices i, j, k can range from 1 to n. This object has an intrinsic meaning, and is called a *tensor*, but we shall not discuss tensors in this set of lecture notes.

We can now state a very important property of functions which are twice continuously differentiable at $x_0 \in A$: the bilinear map $D^2f(x_0)$ is symmetric, meaning that

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \text{for each} \quad i, j \in \{1, ..., n\} \,.$$

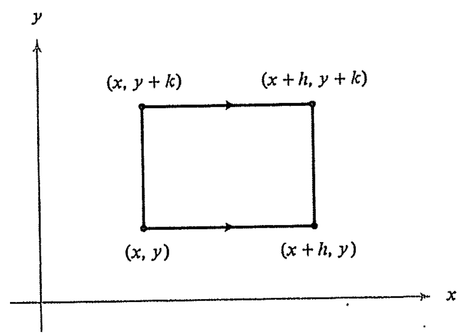
Theorem 2.44 ($D^2f(x_0)$ is symmetric)**.** *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ be twice differentiable on the open set A with D^2f continuous (meaning that all of the components $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are continuous). Then $D^2f(x)$ is symmetric for each $x \in A$ so that*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad \textit{for each} \quad i, j \in \{1, ..., n\} \,. \tag{2.10}$$

Proof. Since we want to prove that (2.10) holds, by fixing all other variables, we can reduce the problem to the case of two independent variables; thus, we assume that $f = f(x, y)$, is a twice continuously differentiable real-valued functions on $A \subset \mathbb{R}^2$.

We form difference quotients about the point (x, y) by perturbing this point a distance h horizontally and a distance k vertically. For (h, k) taken small enough (meaning that $\|(h, k)\| = \sqrt{h^2 + k^2}$ is small enough) so that $(x, y) + (h, k) \in A$, we consider

$$S_{h,k} = [f(x + h, y + k) - f(x, y + k)] - [f(x + h, y) - f(x, y)] . \tag{2.11}$$



We next define the function g_k

$$g_k(u) = f(u, y + k) - f(u, y) ,$$

and note that $S_{h,k}$ can be written in terms of g_k as

$$S_{h,k} = g_k(x + h) - g_k(x) .$$

It follows from the mean value theorem that

$$S_{h,k} = \frac{dg_k}{dx}(c_{k,h}) \cdot h \quad \text{for some} \quad c_{h,k} \in (x, x + h) .$$

Hence,

$$\begin{aligned} S_{h,k} &= \left[\frac{\partial f}{\partial x}(c_{h,k}, y + k) - \frac{\partial f}{\partial x}(c_{h,k}, y)\right] \cdot h \\ &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x}(c_{h,k}, d_{h,k}) \cdot hk \quad \text{for some} \quad d_{h,k} \in (y, y + k) , \end{aligned}$$

where we have once again employed the mean value theorem.

Since $S_{h,k}$ is symmetric with respect to both h, k and x, y, by interchanging the two middle terms in (2.11), we can derive in the identical fashion that

$$S_{h,k} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(\tilde{c}_{h,k}, \tilde{d}_{h,k}) \cdot hk .$$

Equating these two formulas (and dividing by hk), we see that

$$\lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial y \partial x}(c_{h,k}, d_{h,k}) = \lim_{(h,k)\to(0,0)} \frac{\partial^2 f}{\partial x \partial y}(\tilde{c}_{h,k}, \tilde{d}_{h,k}) .$$

Since the second derivatives of f are assumed continuous, we find that

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \frac{\partial^2 f}{\partial x \partial y}(x, y) .$$

$\square$

By a simple induction argument, it can be proven that all the higher-order derivatives of f are symmetric as well. Furthermore, the case that $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ can now be treated also, by applying the above definition to each component of f.

The symmetry of second derivatives represents a fundamental property not encountered in single variable calculus. Examples may shed some light on this.

Example 2.45. *Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is given by $f(x, y, z) = e^{xy} \sin x + x^2 y^4 \cos^2 z$. Then*

$$\frac{\partial f}{\partial x} = e^{xy} \cos c + y e^{xy} \sin x + 2xy^4 \cos^2 z ,$$
$$\frac{\partial f}{\partial y} = x e^{xy} \sin x + 4x^2 y^3 \cos^2 z ,$$

and

$$\frac{\partial^2 f}{\partial y \partial x} = x e^{xy} \cos x + e^{xy} \sin x + xy e^{xy} \sin x + 8xy^3 \cos^2 z$$

which is the same as $\frac{\partial^2 f}{\partial x \partial y}$.

Example 2.46. *Now suppose that $f(x, y) = yx^2 \cos(y^2)$. Then*

$$\frac{\partial f}{\partial x} = 2xy \cos(y^2) , \quad \frac{\partial^2 f}{\partial y \partial x} = 2x \cos(y^2) - 4xy^2 \sin(y^2) ,$$

while

$$\frac{\partial f}{\partial y} = x^2 \cos(y^2) - 2y^2 x^2 \sin(y^2) , \quad \frac{\partial^2 f}{\partial x \partial y} = 2x \cos(y^2) - 4xy^2 \sin(y^2) .$$

As usual, we denote by $A \subset \mathbb{R}^n$ an open subset.

Definition 2.47 (k-times continuously differentiable functions). *A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be of class C^k, $k \geq 0$ integer, and denoted by*

$$f \in C^k(A; \mathbb{R}^m)\,,$$

if the first k derivatives of f exist and are continuous on A. When f is real-valued, we often write $f \in C^k(A)$ instead of $f \in C^k(A; \mathbb{R})$. In this case, we write $f \in C^\infty(A; \mathbb{R}^m)$.

Definition 2.48 (k-times continuously differentiable and uniformly bounded functions). *When the closed set $\overline{A}$ is compact, we write*

$$f \in C^k(\overline{A}; \mathbb{R}^m)$$

to denote the k-times continuously differentiable and uniformly bounded functions on the compact set $\overline{A}$. Similarly, $C^\infty(\overline{A}; \mathbb{R}^m)$ denotes the space of smooth, uniformly bounded functions on $\overline{A}$.

Definition 2.49 (Smooth functions). *A function $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$is called smooth or of class C^∞ if $f \in C^k(A; \mathbb{R}^m)$ for all $k \geq 0$.*

2.11 Taylor's theorem

We continue to let $A \subset \mathbb{R}^n$ denote an open set.

Theorem 2.50 (Taylor's theorem). *Suppose that $f \in C^k(A)$, and let $x, y \in A$ such that the segment joining x and y is in A. Then there exists a point c on that segment such that*

$$f(y) = f(x) + \sum_{l=1}^{k-1} \frac{1}{l!} D^l f(x) \underbrace{(y-x,\ \ldots, y-x)}_{l \text{ vectors}} + \frac{1}{k!} D^k f(c) \underbrace{(y-x,\ \ldots, y-x)}_{k \text{ vectors}}\,,$$

where $D^l f(x)(y-x,\ \ldots, y-x)$ denotes $D^l f(x)$ as l-linear map applied to the l-tuple $(y-x,\ \ldots, y-x)$.

In coordinates,

$$D^l f(x)(y-x,\ \ldots, y-x) = \sum_{i_1,\ldots,i_l=1}^{n} \frac{\partial^l f}{\partial x_{i_1} \cdots \partial x_{i_l}} (y_{i_1} - x_{i_1}) \cdots (y_{i_l} - x_{i_l})$$

Setting $y = x + h$, we can write the Taylor formula as

$$f(x+h) = f(x) + Df(x) \cdot h + \cdots + \frac{1}{(k-1)!} D^{k-1} f(x) \cdot (h, ..., h) + R_{k-1}(x, h)\,,$$

where $R_{k-1}(x,h)$ is the remainder and satisfies

$$\frac{R_{k-1}(x,h)}{\|h\|^{k-1}} \to 0 \text{ as } h \to 0\,.$$

Lagrange's form of the remainder is given by

$$R_{k-1}(x,h) = \sum_{i_1,\ldots,i_k=1}^{n} \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(x+th) h_{i_1} \cdots h_{i_k}\, dt\,.$$

In the course of proving this theorem, we will see other equivalent forms of the remainder term. Note well that this theorem is a generalization of the mean value theorem (which is the case that $k=1$), and of Taylor's theorem for functions of one variable that is normally covered in lower-division calculus courses.

From Taylor's theorem, we are led to form the *Taylor series* of f about the point x_0:

$$\sum_{l=0}^{\infty} D^l f(x_0) \underbrace{(x-x_0,\ \ldots, x-x_0)}_{l \text{ vectors}}$$

where we use the usual convention that $D^0 f(x_0) = f(x_0)$. This series does not always converge, even when f is C^∞; when the series does converge in some neighborhood of x_0, we say the function f is *real analytic* at x_0. To show that a function f is real analytic (and hence that the Taylor series converges in a neighborhood of x_0) amounts to showing that the remainder term

$$\frac{1}{k!} D^k f(c)(y-x,\ \ldots, y-x) \to 0 \quad \text{as} \quad k \to \infty\,.$$

This then is used to establish the usual power series expressions for $\sin x$, $\cos x$, $\exp x$, and so forth.

Example 2.51. *Suppose that $f \in C^\infty([a,b])$ for every closed interval $([a,b]) \subset \mathbb{R}$, and that there exists a constant M such that $|f^{(l)}(x)| \le M$ for all integers $l \ge 0$ and for all $x \in [a,b]$. Then f is real analytic at each x_0 and for $|x-x_0|<1$,*

$$f(x) = \sum_{l=0}^{\infty} \frac{f^{(l)}(x_0)}{l!}(x-x_0)^l\,.$$

To see this, we estimate the remainder:

$$\left| \frac{f^{(l)}(x_0)}{l!}(x-x_0)^l \right| \le \frac{M|x-x_0|^l}{l!} \to 0 \quad \textit{as} \quad l \to \infty\,.$$

Example 2.52. *The function*

$$f(x) = \begin{cases} 0, & x \le 0 \\ e^{-1/x}, & x > 0 \end{cases}$$

is $C^\infty(\mathbb{R})$ but is not real analytic. To see that f is indeed smooth, we must check its behavior at $x = 0$. Notice that for $x > 0$,

$$f'(x) = \frac{1}{x^2} e^{-1/x}$$

which converges to zero as $x \to 0^+$ (by l'Hospital's rule for example (and a change of variables $y = 1/x$)). Similarly, we see that $f^{(l)}(x) \to 0$ as $x \to 0^+$. Thus, $f^{(l)}(0) = 0$ and $f \in C^l(\mathbb{R})$ for each $l \ge 0$, so that $f \in C^\infty(\mathbb{R})$. But since each derivative vanishes at $x = 0$, the Taylor series is identically zero in a neighborhood of $x = 0$, which is not equal to f. It follows that f is not real analytic.

Example 2.53. *Let us compute the second-order Taylor expansion for $f(x, y) = \sin(x+2y)$ in a neighborhood of $(x, y) = (0, 0)$. We first compute $f(0, 0) = 0$. Next, we have that*

$$\begin{aligned}
&\frac{\partial f}{\partial x}(0,0) = \cos(0 + 2 \cdot 0) = 1\,, \\
&\frac{\partial f}{\partial y}(0,0) = 2\cos(0 + 2 \cdot 0) = 2\,, \\
&\frac{\partial^2 f}{\partial^2 x}(0,0) = 0\,, \quad \frac{\partial^2 f}{\partial^2 y}(0,0) = 0\,, \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0\,.
\end{aligned}$$

It follows for (h, k) close to $(0, 0)$, we have that

$$f(h, k) = h + 2k + R_2(h, k)$$

where

$$\frac{R_2(h,k)}{|(h,k)|^2} \to 0 \quad \text{as} \quad (h,k) \to (0,0)\,.$$

Proof of Theorem 2.50. By the chain-rule,

$$\frac{d}{dt} f(x + th) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x + th) h_i\,,$$

and thanks to the fundamental theorem of calculus, we write

$$f(x + h) - f(x) = \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x + th) h_i\, dt\,.$$

The idea is to employ our integration-by-parts Theorem 1.39 to the right-hand side of this expression. Using linearity of the integral and the fact that $-\frac{d}{dt}(1-t) = 1$, we write

$$\int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x+th)h_i\, dt = -\sum_{i=1}^n \int_0^1 \frac{d}{dt}(1-t)\frac{\partial f}{\partial x_i}(x+th)h_i\, dt$$

and integrate-by-parts with respect to $\frac{d}{dt}$:

$$\int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x+th)h_i\, dt = \sum_{i,j=1}^n \int_0^1 (1-t)\frac{\partial^2 f}{\partial x_i \partial x_j}(x+th)h_i h_j\, dt + \frac{\partial f}{\partial x_i}(x)h_i\,.$$

We have thus shown that

$$f(x+h) - f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)h_i + R_1(x,h)\,,$$

where

$$R_1(x,h) = \sum_{i,j=1}^n \int_0^1 (1-t)\frac{\partial^2 f}{\partial x_i \partial x_j}(x+th)h_i h_j\, dt\,. \tag{2.12}$$

Since each component of h satisfies $|h_i| \le \|h\|$, we have that

$$|R_1(x,h)| \le \|h\|^2 \left[\sum_{i,j=1}^n \int_0^1 (1-t)\left|\frac{\partial^2 f}{\partial x_i \partial x_j}(x+th)\right| dt \right] \le \|h\|^2 M_1$$

for some constant $M_1 > 0$, since the integrand is bounded on the segment between x and $x+h$.

If instead of estimating R_1, we continue the analysis by further integration-by-parts in (2.12), then we find that

$$R_1(x,h) = \sum_{i,j,k=1}^n \int_0^1 \frac{(1-t)^2}{2} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(x+th)h_i h_j h_k dt + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(x)h_i h_j\,,$$

from which it follows that

$$f(x+h) = f(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)h_i + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x)h_i h_j + R_2(x,h)\,,$$

where

$$R_2(x,h) = \sum_{i,j,k=1}^n \int_0^1 \frac{(1-t)^2}{2} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(x+th)h_i h_j h_k dt\,.$$

Now the integrand is again bounded on the segment between x and $x+h$ so for some constant $M_2 > 0$, we see that

$$|R_2(x,h)| \le \|h\|^3 M_2 \,.$$

In particular, note that $|R_2(x,h)|/\|h\|^2 \le \|h\| M_2 \to 0$ as $h \to 0$.

The other form of the remainder $R_{k-1}(x,h)$ which generalizes the mean value theorem, arises from the so-called second mean value theorem of integrals: there exists $cin(a,b)$ such that

$$\int_a^b u(t)v(t)dt = u(c)\int_a^b v(t)dt$$

provided that u and v are continuous on $[a,b]$ and $v \ge 0$. Returning to the Lagrange form of $R_1(x,h)$, we see that

$$\begin{aligned} R_1(x,h) &= \sum_{i,j=1}^n \int_0^1 (1-t)\frac{\partial^2 f}{\partial x_i \partial x_j}(x+th)h_i h_j \, dt \\ &= \sum_{i,j=1}^n \int_0^1 (1-t) D^2 f(x+th)(h,h)\, dt \\ &= \frac{1}{2} D^2 f(c)(h,h)\,, \end{aligned}$$

where $(1-t)$ played the role of $v(t)$ and $D^2 f$ played the role of $u(t)$ in the second mean value theorem for integrals stated above. Here c is some point on the segment between x and $y = x+h$.

One can proceed via induction to obtain the formula for the remainder at all orders, and this then completes the proof. $\square$

Remark 2.54. *In fact, with more effort, a stronger theorem can be established: if $f \in C^k(A)$, then*

$$f(x+h) = f(x) + \sum_{l=1}^k \frac{1}{l!} D^l f(x) \cdot \underbrace{(h,...,h)}_{k \text{ copies}} + R_k(x,h)\,,$$

where $R_k(x,h)/\|h\|^k \to 0$ as $h \to 0$, $h \in \mathbb{R}$. We leave the proof to the interested student.

2.12 The minima and maxima of functions $f : \mathbb{R}^n \to \mathbb{R}$

Taylor's Theorem 2.50 provides a convenient method to determine the local maxima and minima of functions $f : A \subset \mathbb{R} \to \mathbb{R}$, and this method involves the second derivative $D^2 f$, the Hessian matrix.

Recall that for real-valued functions of a single variable $f : \mathbb{R} \to \mathbb{R}$, if f has a local maximum or minimum at a point x_0 and if f differentiable at x_0, then $f'(x_0) = 0$; furthermore, if f is twice continuously differentiable and if $f''(x_0) < 0$, then x_0 is a local maximum and if $f''(x_0) > 0$, it is a local minimum.

We want to generalize these facts to functions $f : A \subset \mathbb{R}^n \to \mathbb{R}$.

Definition 2.55 (Local minimum/maximum). *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ with A open. If there is a neighborhood U of $x_0 \in A$ on which $f(x_0)$ is a maximum, i.e., $f(x_0) \geq f(x)$ for all $x \in U$, we say that $f(x_0)$ is a local maximum for f. Similarly, we can define the local minimum of f. A point is called extreme if it is either a local minimum or a local maximum for f. A point x_0 is a critical point if f is differentiable at x_0 and $Df(x_0) = 0$.*

Theorem 2.56 (Critical Point). *If $f : A \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable, A is open, and if $x_0 \in A$ is an extreme point of f, then $Df(x_0) = 0$; that is, x_0 is a critical point.*

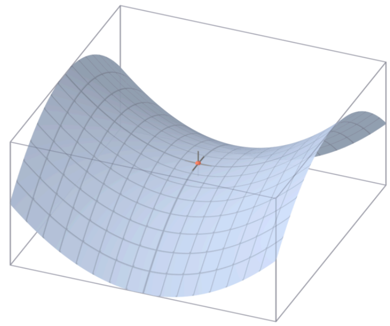
Corollary 2.57 (Multidimensional version of Rolle's theorem). *With $B = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, let $f : B \subset \mathbb{R}^n \to \mathbb{R}$ be continuous and let f be differentiable on int(B), the interior of B. Suppose that $f(x) = 0$ for all $x \in \partial B$, the boundary of B. Then there is a point $x_0 \in$ int(B) for which $Df(x_0) = 0$.*

Proof. If f is identically zero, then the corollary trivially holds, so suppose that $f(x) \neq 0$ for some $x \in \text{int}(B)$. Then f attains a maximum or a minimum at some interior point, since B is compact. Thus there is an extreme point $x_0 \in \text{int}(B)$ and hence by Theorem 2.56, $Df(x_0) = 0$. □

The proof of Theorem 2.56 (given below) is the same as for functions of one variable, and the result is intuitively obvious since at an extreme point, the graph of f must have a horizontal tangent plane. However, if x_0 is a critical point of a function f, this is not sufficient to guarantee that x_0 is also extreme. A simple example is the cubic polynomial $f(x) = x^3$; since $f'(0) = 0$, $x_0 = 0$ is a critical point, but $x^3 > 0$ for $x > 0$ and $x^3 < 0$ for $x < 0$, so 0 is not an extreme point. Another example is $f(x, y) = y^2 - x^2$, for which the origin $(0, 0)$ is a critical point since $Df = (\partial f/\partial x, \partial f/\partial y) = (-2x, 2y)$. However, in any neighborhood of the origin, we can find points where $f > 0$ and points where $f < 0$. A critical point which is not a local extreme value is called a *saddle point* (see Figure below).

Returning to the simple case of $f : A \subset \mathbb{R} \to \mathbb{R}$, we have already reviewed that $f(x_0)$ is a local maximum if $f'(x_0) = 0$ and $f''(x_0) < 0$. Recall that the geometric picture for $f''(x_0) < 0$ is that f is concave downwards. To generalize this concept to $f : A \subset \mathbb{R}^n \to \mathbb{R}$, we define the Hessian of f at x_0.

Definition 2.58 (Hessian). *If $f \in C^2(A)$, the Hessian of f at $x_0 \in A$ is $D^2 f(x_0)$. In the standard basis of $\mathbb{R}^n$, it is the $n \times n$ matrix of partial derivatives $\partial^2 f/\partial x_i \partial x_j$. We view $D^2 f(x_0)$ as a bilinear map taking $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$.*



Definition 2.59. *A bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called positive definite if $B(x, x) > 0$ for all $x \neq 0$ in $\mathbb{R}^n$ and is positive semi-definite if $B(x, x) \geq 0$ for all $x \in \mathbb{R}^n$. Negative definite and negative semi-definite bilinear maps are defined similarly, by reversing the inequality.*

Theorem 2.60. *[Criteria for local maximum/minimum]*

(i) If $f \in C^2(A)$ and $x_0 \in A$ is a critical point of f such that the Hessian $D^2 f(x_0)$ is negative definite, then f has a local maximum at x_0.

(ii) If $f \in C^2(A)$ has a local maximum at $x_0 \in A$, then the Hessian $D^2 f(x_0)$ is negative semi-definite.

The case of a local minimum for f at x_0 is obtained by changing negative to positive. (Note that a minimum of f is a maximum of $-f$.)

When $f : A \subset \mathbb{R}^2 \to \mathbb{R}$, the Hessian is a 2×2 matrix, and in this case it is easy to determine when the Hessian is positive definite (or negative definite).

Lemma 2.61. *The matrix*

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

is positive definite iff $a > 0$ and $ad - b^2 > 0$.

Proof. Positive definite means that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} > 0 \quad \text{if} \quad (x, y) \neq (0, 0)$$

which reduces to the condition $ax^2 + 2bxy + dy^2 > 0$. First, suppose that this inequality holds for all $(x, y) \neq (0, 0)$. Setting $x = 1$ and $y = 0$ shows that $a > 0$. Setting $y = 1$, we see that $ax^2 + 2bx + d > 0$ for all x. This function is a parabola with a minimum (since $a > 0$) at $2ax + 2b = 0$, that is, $x = -b/a$. Hence

$$a\left(-\frac{b}{a}\right)^2 + 2b\left(-\frac{b}{a}\right) + d > 0$$

so that $ad - b^2 > 0$. The converse is proved in the same way. □

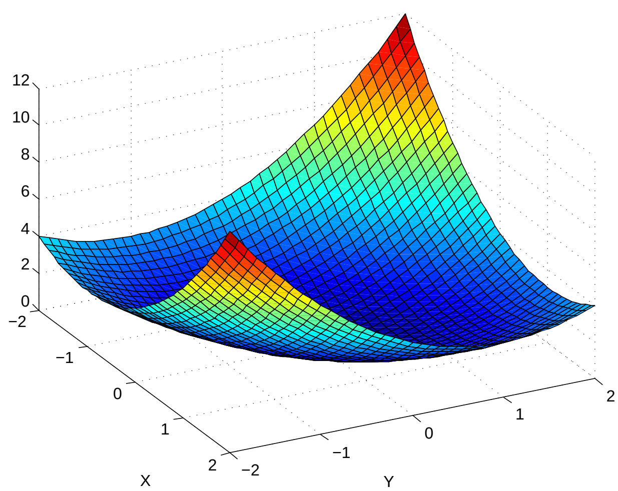
Similarly, we have that

Lemma 2.62. *The matrix*

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

is negative definite iff $a < 0$ and $ad - b^2 > 0$.

Example 2.63. *Consider the function $f(x, y) = x^2 - xy + y^2$. This function is smooth so*



we can compute partial derivatives:

$$\frac{\partial f}{\partial x}(x, y) = 2x - y\,, \quad \frac{\partial f}{\partial y}(x, y) = -x + 2y\,,$$
$$\frac{\partial^2 f}{\partial x^2}(x, y) = 2\,, \frac{\partial^2 f}{\partial y^2}(x, y) = 2\,, \frac{\partial^2 f}{\partial x \partial y}(x, y) = -1\,.$$

The critical point satisfies $2x = y$ and $x = 2y$ and hence must be at $(0, 0)$. The Hessian of f at $(0, 0)$ is given by

$$D^2 f(0, 0) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Since the conditions of Lemma 2.61 are satisfied, then $D^2 f(0,0)$ is positive definite so that $(0,0)$ is a local minimum.

Note that we can compute the eigenvalues λ_i and eigenvectors w_i of the Hessian satisfying:

$$D^2 f(0,0) \cdot w_i = \lambda_i \, w_i \quad \textit{for} \quad i = 1, 2 \, .$$

We find that

$$\lambda_1 = 1 \;\; \textit{and} \;\; w_1 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \\ -\frac{\sqrt{2}}{2} \end{pmatrix} ,$$

and

$$\lambda_2 = 3 \;\; \textit{and} \;\; w_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \\ \frac{\sqrt{2}}{2} \end{pmatrix} .$$

The geometric significance of the Hessian follows from the fact that the eigenvectors w_1 and w_2 are the two principle directions of curvature for the graph(f), while the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 3$ are the two principle curvatures. If both eigenvalues are positive at the critical point x_0, then x_0 is a local minimum, while if both eigenvalues are negative, then we have a local maximum. If one eigenvalue is positive and the other is negative, then x_0 is a saddle point.

Proof of Theorem 2.56. Suppose for the sake of contradiction that $Df(x_0) \neq 0$. In this case, there exists $x \in \mathbb{R}^n$ such that $Df(x_0) \cdot x \neq 0$ and for concreteness, we suppose that for some $c > 0$, $Df(x_0) \cdot x = c$.

Then, we can choose $\delta > 0$ sufficiently small so that whenever $\|h\| < \delta$,

$$\|f(x_0 + h) - f(x_0) - Df(x_0) \cdot h\| \leq \frac{c}{2\|x\|} \|h\| \, .$$

We choose $\lambda > 0$ so that $\lambda \|x\| < \delta$ so that

$$\|f(x_0 + \lambda x) - f(x_0) - Df(x_0) \cdot \lambda x\| \leq \frac{\lambda c}{2} \, .$$

Since $Df(x_0) \cdot \lambda x = \lambda c$,

$$\|f(x_0 + \lambda x) - f(x_0) - \lambda c\| \leq \frac{\lambda c}{2} \, .$$

Thus,

$$f(x_0 + \lambda x) - f(x_0) > 0$$

so $f(x_0)$ cannot be a local maximum.

Similarly, by considering $-(\lambda x)$ instead of λx, we see that

$$\|f(x_0 - \lambda x) - f(x_0) + \lambda c\| \le \frac{\lambda c}{2}$$

from which we conclude that

$$f(x_0 - \lambda x) - f(x_0) < 0\,,$$

so that $f(x_0)$ cannot be a local minimum, and hence if $Df(x_0) \neq 0$, then $f(x_0)$ cannot be an extreme point. $\square$

Example 2.64. *Consider $f(x,y) = x^3 - 3x^2 + y^2$. We compute*

$$\frac{\partial f}{\partial x} = 3x^2 - 6x = 0\,, \quad \frac{\partial f}{\partial y} = 2y = 0\,,$$

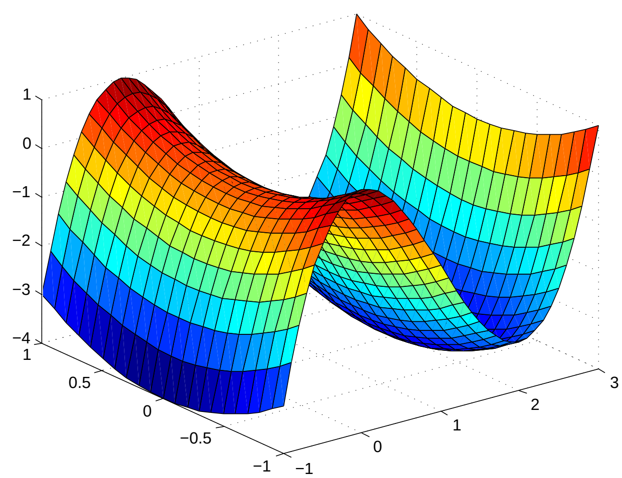
so that the critical points are at $(0,0)$ and $(2,0)$. Since

$$D^2 f = \begin{pmatrix} 6x-6 & 0 \\ 0 & 2 \end{pmatrix}\,,$$

we see that

$$D^2 f(0,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} \quad \textit{and} \quad D^2 f(2,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

It follows that $(2,0)$ is a local minimum and $(0,0)$ is a saddle point.



Proof of Theorem 2.60. (i) If $D^2f(x_0)$ is negative definite, then $D^2f(x_0)(x,x) < 0$ for all $x \neq 0$ in $\mathbb{R}^n$. As $D^2f(x_0)(x,x)$ is blinear in x and since linear maps are continuous, it follows that $D^2f(x_0)(x,x)$ is a continuous function of $x \in \mathbb{R}^n$.

Let $\mathbb{S} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. $\mathbb{S}$ is compact, so there is some point $\bar{x} \in \mathbb{S}$ such that

$$D^2f(x_0)(x,x) \leq D^2f(x_0)(\bar{x},\bar{x}) < 0 \quad \text{for all} \quad x \in \mathbb{S}\,.$$

Now, let $0 < \epsilon = D^2f(x_0)(\bar{x},\bar{x})$. Then, since $D^2f(x_0)$ is bilinear,

$$\|x\|^2 D^2f(x_0)\left(\frac{x}{\|x\|}, \frac{x}{\|x\|}\right) \leq -\epsilon\|x\|^2 \quad \forall x \in \mathbb{R}^n, x \neq 0\,,$$

where we have used the fact that $x/\|x\| \in \mathbb{S}$ whenever $x \neq 0$. Since D^2f is continuous on A, there exists $\delta > 0$ such that $\|y - x_0\| < \delta$ implies that $\|D^2f(y) - D^2f(x_0)\| < \bar{\epsilon}/2$. Here $\delta > 0$ is chosen so small so as to ensure that $y \in A$. Also, for each $x \in A$, $D^2f(x)$ is an $n \times n$ matrix so its norm is given by $\|D^2f(x)\|^2 = \sum_{i,j=1}^n \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]^2$. Since $|D^2f(x_0)(x,x)| \leq C\|D^2f(x_0)\|\|x\|^2$ for some constant $C > 0$ depending only on n, we set $\epsilon = C\bar{\epsilon}$.

Now, by Taylor's Theorem 2.50, there is a point c on the segment between x_0 and y in A such that

$$f(y) - f(x_0) = Df(x_0) \cdot (y - x) + \frac{1}{2}D^2f(c)(y - x_0, y - x_0)\,.$$

We write

$$D^2f(c)(y-x_0,y-x_0) = D^2f(x_0)(y-x_0,y-x_0)+D^2f(c)(y-x_0,y-x_0)-D^2f(x_0)(y-x_0,y-x_0)$$

Since $\|D^2f(c) - D^2f(x_0)\| < \epsilon/2$, we see that

$$\begin{aligned} D^2f(c)(y - x_0, y - x_0) &\leq D^2f(x_0)(y - x_0, y - x_0) \\ &\qquad + |D^2f(c)(y - x_0, y - x_0) - D^2f(x_0)(y - x_0, y - x_0)| \\ &\leq -\epsilon\|y - x_0\|^2 + \frac{\epsilon}{2}\|y - x_0\|^2 \\ &= -\frac{\epsilon}{2}\|y - x_0\|^2\,. \end{aligned}$$

Since $Df(x_0) = 0$, we find that

$$f(y) - f(x_0) = \frac{1}{2}D^2f(c)(y - x_0, y - x_0) \leq -\frac{\epsilon}{4}\|y - x_0\|^2 < 0\,.$$

It follows that $f(y) \leq f(x_0)$ for all points y such that $\|y - x_0\| < \delta$, and so f has a local maximum at x_0.

(ii) For the sake of contradiction, suppose that f has a local maximum at x_0 and that $D^2f(x_0)(x,x) > 0$ for some $x \in \mathbb{R}^n$. We define $g(t) = -f(x_0 + tx)$. Since f is defined in a neighborhood of x_0, then g is defined in a neighborhood of $t = 0$. By the chain-rule,

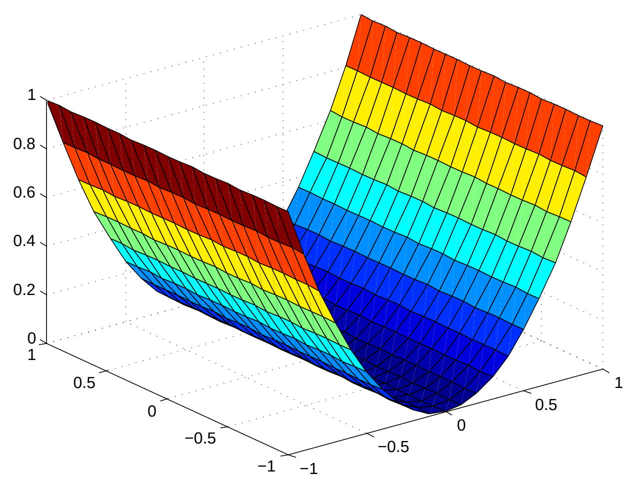
$$g'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0 + tx)x_i \text{ and } g''(t) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0 + tx)x_i x_j \,,$$

so that

$$g'(0) = Df(x_0) \cdot x = 0\,, \text{ and } g''(0) = -D^2f(x_0)(x,x) < 0\,.$$

By part (i), g has a local maximum at $t = 0$, so that for $|t| < \delta$, $g(t) < g(0)$ for $t \neq 0$. This, in turn, implies that $f(x_0+tx) > f(x_0)$ for $|t| < \delta$ so that f does not have a local maximum at x_0, and we have reached a contradiction. Thus, $D^2f(x_0)(x,x) \leq 0$ for all $x \in \mathbb{R}^n$. $\square$

As can be seen from this proof, Theorem 2.60 only guarantees semi-definiteness of the Hessian at a local minimum and maximum, and the question that was asked in class was what type of function has a local minimum (for example) at x_0 for which $D^2f(x_0)(x,x) = 0$ for some $x \in \mathbb{R}^n$. The geometric interpretation of the Hessian provides the answer, for suppose that we have a function $f(x,y)$ which has curvature 0 in one direction and positive curvature in another direction. The function $f(x,y) = x^2$ does the job:



2.13 Exercises

Problem 2.1. *Compute $Df(x)$ for $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x \sin x$.*

Problem 2.2. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are both differentiable. Prove that $D(f+g) = Df + Dg$.*

Problem 2.3. *Let $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1\,, x_2 = 0\}$. This is not an open set. Prove that the conclusion of Theorem 2.5 is false for this set A. (Hint. Take, for example, $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x_1, x_2) = 0$ and show that both $Df(x_1, x_2) = 0$ and $Df(x_1, x_2)\cdot(w_1, w_2) = w_2$ satisfy the definition of derivative.)*

Problem 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and suppose that there is a constant M such that for $x \in \mathbb{R}^n$, $\|f(x)\| \le M\|x\|^2$. Prove that f is differentiable at $x_0 = 0$ and that $Df(x_0) = 0$.*

Problem 2.5. *If $f : \mathbb{R} \to \mathbb{R}$ and $|f(x)| \le |x|$, must $Df(0) = 0$?*

Problem 2.6. *Does the mean value theorem (Theorem 2.9) apply to the function $f(x) = \sqrt{x}$ on $[0, 1]$? Does it apply to $g(x) = \sqrt{|x|}$ on $[-1, 1]$?*

Problem 2.7. *Is the Lipschitz condition (2.4) strong enough to guarantee differentiability?*

Problem 2.8. *Let $f(x) = x \sin(1/x)$ for $x \ne 0$ and $f(0) = 0$. Investigate the continuity and differentiability of $f(x)$ at $x = 0$.*

Problem 2.9. *Use Theorem 2.22 to show that*

$$f(x,y) = \begin{cases} \frac{(xy)^2}{\sqrt{x^2+y^2}}, & (x,y) \ne (0,0), \\ 0, & (x,y) = (0,0)\,. \end{cases}$$

is differentiable at $(x, y) = (0, 0)$.

Problem 2.10. *Let $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$. Is $f(x, y)$ differentiable at $(x, y) = (0, 0)$ if $f(0, 0) = 0$?*

Problem 2.11. *Find the unit normal to the surface $x^2 - y^2 + xyz = 1$ at the point $(1, 0, 1)$.*

Problem 2.12. *In what direction is $f(x, y) = e^{x^2 y}$ increasing the fastest?*

Problem 2.13. *Prove l'Hospital's rule: if f', g' exist at x_0, $g'(x_0) \ne 0$, and if $f(x_0) = 0 = g(x_0)$, then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}\,.$$

Problem 2.14. *Using Problem 2.13, evaluate*

(a) $\lim_{x\to 0} \frac{\sin x}{x}$.

(b) $\lim_{x\to 0} \frac{e^x - 1}{x}$.

Problem 2.15. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable with A convex and suppose that $\|\operatorname{grad} f(x)\| \leq M$ for $x \in A$. Prove that*

$$|f(x) - f(y)| \leq M\|x - y\| \quad \textit{for} \quad x, y \in A\,.$$

Do you think that this is true if A is not convex?

Problem 2.16. *Let*

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right)\,, & x \in (-1, 0) \cup (0, 1)\,, \\ 0\,, & x = 0\,. \end{cases}$$

How can you apply Taylor's theorem to f about the point $x = 0$?

Problem 2.17. *Find the Taylor series representation about $x = 0$ for $f(x) = \log(1 - x)$, $x \in (-1, 1)$ and show that this series expansion equal $f(x)$ for each $x \in (-1, 1)$. Also, show that this series converges uniformly to f on all closed subintervals of $(-1, 1)$.*

Problem 2.18. *Verify that if the conditions in Example 2.51 are met then we can differentiate the Taylor series term by term to obtain $f'(x)$.*

Problem 2.19. *Investigate the nature of the critical point $(0, 0)$ of $f(x, y) = x^2 + 2xy + y^2 + 6$.*

Problem 2.20. *Determine the nature of the critical point $(0, 0)$ of $f(x, y) = x^3 + 2xy^2 - y^4 + x^2 + 3xy + y^2 + 10$.*

3 Inverse and Implicit Function Theorems

Given a system of linear equations,

$$\begin{array}{ccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = y_1 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & \cdot & & \cdot \\ a_{n1}x_1 & + & \cdots & + & a_{nn}x_n & = y_1 \end{array}$$

the vector $(x_1, . . ., x_n)$ of unknowns can be solved for, whenever the matrix $A = (a_{ij})$ is non-singular, i.e. $\det A \neq 0$. The question that we would like to examine is the following: given a system of nonlinear equations

$$\begin{array}{ccc} f_1(x_1, ..., x_n) & = & y_1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ f_n(x_1, ..., x_n) & = & y_n \end{array}\,,$$

under what conditions can we solve for the vector $(x_1, \ldots, x_n)$ of unknowns? Here, f should be viewed as the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ and the object that plays the role of $\det A$ for the linear system is the *Jacobian determinant* of f:

$$\det Df(x) = \det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} .$$

It turns out that whenever $\det Df(x) \neq 0$, we might indeed be able to solve the vector equation $f(x) = y$ for x. The inverse function theorem will provide us with the necessary conditions for such a solution to exist. On the other hand, if we write $F(x, y) = f(x) - y$, then we may wish to find the pair (x, y) which solves $F(x, y) = 0$ with y an *implicit function* of x, and it is the implicit function theorem which describes those circumstances when such an implicit solution exists.

3.1 The space of continuous functions

Let $A \subset \mathbb{R}^n$ and let $\mathcal{V}$ denote the set of functions $f : A \to \mathbb{R}^m$. Then $\mathcal{V}$ is a vector space, the zero element is the function $f(x) = 0$ for all $x \in A$, and $(f + g)(x) = f(x) + g(x)$, $(\lambda f)(x) = \lambda f(x)$ for each $\lambda \in \mathbb{R}$, $f, g \in \mathcal{V}$. We let

$$C(A; \mathbb{R}^m) = \{ f : A \to \mathbb{R}^m \mid f \text{ is continuous} \} .$$

Then $C(A; \mathbb{R}^m)$ is also a vector space since the sum of two continuous functions is continuous, and for each $\lambda \in \mathbb{R}$ and $f \in C(A; \mathbb{R}^m)$, $\lambda f \in C(A; \mathbb{R}^m)$.

Next, let $C_b(A; \mathbb{R}^m$ denote the vector subspace of $C(A; \mathbb{R}^m)$ consisting of bounded functions:

$$C_b(A; \mathbb{R}^m) = \{ f \in C(A; \mathbb{R}^m) \mid \|f\| \leq M \}$$

for some constant $M < \infty$. In the case that A is closed and compact, then $C(A; \mathbb{R}^m) = C_b(A; \mathbb{R}^m)$.

Definition 3.1 (Norm on $C_b(A; \mathbb{R}^m)$)**.** *For $f \in C_b(A; \mathbb{R}^m)$, let*

$$\|f\|_{C_b(A;\mathbb{R}^m)} = \sup_{x \in A} \|f(x)\|$$

which exists since f is bounded. The real number $\|f\|_{C_b(A;\mathbb{R}^m)}$ is called the norm of f .

The norm $\|f\|_{C_b(A;\mathbb{R}^m)}$ is what Rosenlicht, Chapter 4, refers to as the largest distance $\sup_{x \in A} d(f(x), 0)$ from $f(x)$ to the origin 0. It is a measure of the size of f, just as $\|x\|$ is a measure of the size of the vector $x \in \mathbb{R}^n$. Note that $\|f\|_{C_b(A;\mathbb{R}^m)} \leq M$ iff $|f(x)| \leq M$ for all $x \in A$.

Theorem 3.2. *The function* $f \mapsto \|f\|_{C_b(A;\mathbb{R}^m)}$ *satisfies the properties of a norm:*

(i) $\|f\|_{C_b(A;\mathbb{R}^m)} \geq 0$ *and* $\|f\|_{C_b(A;\mathbb{R}^m)} = 0$ *iff* $f = 0$*;*

(ii) $\|\alpha f\|_{C_b(A;\mathbb{R}^m)} = |\alpha| \|f\|_{C_b(A;\mathbb{R}^m)}$ *for all* $\alpha \in \mathbb{R}$, $f \in C_b(A;\mathbb{R}^m)$*;*

(iii) $\|f+g\|_{C_b(A;\mathbb{R}^m)} \leq \|f\|_{C_b(A;\mathbb{R}^m)} + \|g\|_{C_b(A;\mathbb{R}^m)}$ *(triangle inequality).*

Given two functions $f, g \in C_b(A;\mathbb{R}^m)$,

$$\|f-g\|_{C_b(A;\mathbb{R}^m)}$$

measures the difference between these two functions. According to (i) of Theorem 3.2, $f = g$ iff $\|f-g\|_{C_b(A;\mathbb{R}^m)} = 0$.

Definition 3.3 (Convergence in $C_b(A;\mathbb{R}^m)$)**.** *Given a sequence* $\{f_n\}$ *in* $C_b(A;\mathbb{R}^m)$*, we say*

$$f_n \to f \text{ in } C_b(A;\mathbb{R}^m) \text{ if } \lim_{n\to\infty} \|f_n - f\|_{C_b(A;\mathbb{R}^m)} = 0\,.$$

When A is compact, this convergence is the same as uniform convergence of $f_n \to f$.

When a metric space is endowed with a norm, the way that $C_b(A;\mathbb{R}^m)$ is, there is a special name for this space.

Definition 3.4 (Banach Space)**.** *A normed vector space which is complete is called a Banach space. Every Cauchy sequence in a Banach space* X *converges to a limit in* X*.*

As is established in the second theorem of Rosenlicht, Chapter 4, Section 6, the space $C_b(A;\mathbb{R}^m)$ is a complete metric space, meaning that all Cauchy sequences converge. We state this as a theorem.

Theorem 3.5. *The normed vector space* $C_b(A;\mathbb{R}^m)$ *is a Banach space.*

Example 3.6. *Let* $B = \{f \in C([0,1];\mathbb{R}) \mid f(x) > 0 \ \forall x \in [0,1]\}$*. We will prove that* B *is an open set in* $C([0,1];\mathbb{R}$*. By definition, for any* $f \in B$*, we must produce an* $\epsilon > 0$ *which is sufficiently small so that* $\{g \in C([0,1];\mathbb{R}) \mid \|f-g\|_{C([0,1];\mathbb{R})}\} < \epsilon$*. Since* $[0,1]$ *is compact,* f *has a minimum value, say* m*, at some point of* $[0,1]$*. Thus* $f(x) \geq m > 0$ *for all* $x \in [0,1]$*. Let* $\epsilon = m/2$*. Then if* $\|f-g\|_{C([0,1];\mathbb{R})} < \epsilon$*, then for any* $x \in [0,1]$*,* $|f(x) - g(x)| < \epsilon = m/2$*. Hence* $g(x) \geq m/2$*, so* $g \in B$*.*

Example 3.7. *The closure of the set* B *in the above example is* $\overline{B} = \{f \in C([0,1];\mathbb{R}) \mid f(x) \geq 0 \ \forall x \in [0,1]\}$*. The set* $\overline{B}$ *is closed because if* $f_n(x) \geq 0$ *and* $f_n \to f$ *uniformly on* $[0,1]$*, then* $f(x) \geq 0$ *for all* $x \in [0,1]$*. To show that* $\overline{B}$ *is indeed the closure of* B*, we must show that for every* $f \in \overline{B}$*, there is a sequence of functions* f_n *in* B *such that* $f_n \to f$ *uniformly. The sequence* f_n *is then chosen to be* $f_n = f + 1/n$*.*

Example 3.8. *Suppose that we have a sequence of functions f_n in $C_b(A;\mathbb{R}^m)$ and that*

$$\|f_n - f\|_{C_b(A;\mathbb{R}^m)} \le r_n\,, \quad \textit{where} \quad \sum_{n=1}^{\infty} r_n \quad \textit{converges}\,.$$

Then f_n converges to some f in $C_b(A;\mathbb{R}^m)$. Indeed, this follows by an application of the triangle inequality, since

$$\begin{aligned}\|f_n - f_{n+k}\|_{C_b(A;\mathbb{R}^m)} &\le \|f_n - f_{n+1}\|_{C_b(A;\mathbb{R}^m)} + \|f_{n+1} - f_{n+2}\|_{C_b(A;\mathbb{R}^m)} \\ &\qquad + \cdots + \|f_{n+k-1} - f_{n+k}\|_{C_b(A;\mathbb{R}^m)} \\ &\le r_n + r_{n+1} + \cdots + r_{n+k}\,.\end{aligned}$$

Since $\sum_{n=1}^{\infty} r_n$ converges, then $\lim_{n\to\infty} \sum_{l=n}^{n+k} r_l = 0$. Hence $\{f_n\}$ is a Cauchy sequence, and hence converges to some f in $C_b(A;\mathbb{R}^m)$ by Theorem 3.5.

3.2 Contraction mapping theorem

Theorem 3.9 (Contraction Mapping Principle)**.** *Let $T : C_b(A;\mathbb{R}^m) \to C_b(A;\mathbb{R}^m)$ be a given mapping such that there exists a constant λ with $0 \le \lambda < 1$ and such that*

$$\|T(f) - T(g)\|_{C_b(A;\mathbb{R}^m)} \le \lambda \|f - g\|_{C_b(A;\mathbb{R}^m)} \quad \forall f, g \in C_b(A;\mathbb{R}^m)\,.$$

Then T (is continuous and) has a unique fixed point; that is, there exists a unique element $f_0 \in C_b(A;\mathbb{R}^m)$ such that $T(f_0) = f_0$.

In fact, the theorem is valid for any complete metric space, and not just $C_b(A;\mathbb{R}^m)$. In such a general metric space, the condition on T would be $d(T(f), T(g)) \le \lambda d(f, g)$. Such a map is called a *contraction* as it shrinks distances between two functions f and g by a factor $\lambda < 1$. The method of proof is called *successive approximations* or sometimes *Newton's method.* We are search for the fixed point f_0 for which $T(f_0) = f_0$, so the method begins with some guess: $f \in C_b(A;\mathbb{R}^m)$. Then we form the sequence

$$f,\ T(f),\ T^2(f) = T(T(f)),\ T^3(f) = T(T(T(f))), \ldots$$

Remarkaby, this sequence is Cauchy with respect to the norm on $C_b(A;\mathbb{R}^m)$, and hence converges to some limit function in $C_b(A;\mathbb{R}^m)$ which necessarily is the fixed-point we were after. This method of successive approximations is constructive, and is immensely useful in many numerical algorithms designed to solve nonlinear equations. One successively computes the elements of the approximating sequence $T^n(f)$, and if by some luck, the fixed-point is obtained at some finite $n < \infty$, then the sequence "stops."

3.3 Proof of the Contraction Mapping Principle

In the event that the previous section skipped, we will give a proof of the contraction mapping theorem in a slightly more general setting.

Theorem 3.10. *Let X denote a complete metric space and let $T : X \to X$ be a contraction: $d(T(f), T(g)) \le \lambda\, d(f, g)$ for all $f, g \in X$, and $0 \le \lambda < 1$ a fixed constant. Then T is continuous and has a unique fixed-point.*

Proof. Uniform continuity of T follows immediately from the contraction property of T; namely for any $\epsilon > 0$, let $\delta = \epsilon/\lambda$, so that $d(f, g) < \delta$ implies that $d(T(f), T(g)) < \lambda\delta = \epsilon$.

Next, let $f_0 \in X$ and set $f_1 = T(f_0)$, $f_2 = T(f_1)$, ..., $f_{n+1} = T(f_n) = T^{n+1}(f_0)$. We claim that f_n is a Cauchy sequence in X. Since

$$
\begin{aligned}
d(f_{n+1}, f_n) &= d(T(f_n), T(f_{n-1})) \\
&\le \lambda d(f_n, f_{n-1}) \\
&= \lambda d(T(f_{n-1}), T(f_{n-2})) \\
&\le \lambda^2 d(f_{n-1}, f_{n-2}) \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&\le \lambda^n d(f_1, f_0) = \lambda^n d(T(f_0), f_0)\,,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
d(f_n, f_{n+k}) &\le d(f_n, f_{n+1}) + d(f_{n+1}, f_{n+2}) + \cdots + d(f_{n+k-1}, f_{n+k}) \\
&\le (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+k-1}) d(T(f_0), f_0)\,.
\end{aligned}
$$

Because $\lambda < 1$, $\sum_{l=1}^{\infty} \lambda^n$ is a convergent geometric series, so given $\epsilon > 0$, there is an N such that $n \ge N$ implies $(\lambda^n + \lambda^{n+1} + \cdots + \lambda^{n+k-1}) < \epsilon/d(T(f_0), f_0)$. Hence $n \ge N$ implies that $d(f_n, f_{n+k}) < \epsilon$. Thus f_n is a Cauchy sequence, and by completeness of X, $f_n \to f$ for some $f \in X$.

It remains to prove that $T(f) = f$ as asserted. Since $f_n \to f$ in X and $T : X \to X$ is continuous, we see that

$$T(f) = \lim_{n\to\infty} T(f_n)\,.$$

On the other hand, since $T(f_n) = f_{n+1}$,

$$T(f) = \lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} f_{n+1} = f\,.$$

To see that f is the unique fixed-point of T, suppose that $T(f) = f$ and $T(g) = g$. Then

$$d(f,g) = d(T(f), T(g)) \leq \lambda\, d(f,g)\,.$$

If $d(f,g) \neq 0$, then $1 \leq \lambda$ which contradicts our assumption. It then must be that $d(f,g) = 0$ or $f = g$. □

3.4 The fundamental theorem of ODEs

One of the most important applications of the Contraction Mapping Theorem (CMT) is the construction of solutions to ordinary differential equations (ODEs). In lower-division ODE courses such at MAT22B, explicit solutions are constructed for certain linear differential equations; for example, the solution to $\frac{dx}{dt} = ax(t)$ is given by $x(t) = Ce^{at}$ and the solution to $\frac{d^2x}{dt^2} + a^2x = 0$ is $x(t) = C\cos(kt - \omega)$ for some constants C and ω. For nonlinear ODEs, it is generally not possible to construct explicit solutions, so a natural question is if these equations always have solutions? If so, are these solutions unique? If so, do the solutions exists for all time (t can be thought of as time in this context) or do they have a finite-time "blowup," meaning that the solution becomes unbounded in finite time.

Example 3.11. *Consider the nonlinear equation $\frac{dx}{dt} = x^2$ for $t > 0$ and $x(0) = 1$. We can directly integrate to find the solution: we write $dx/x^2 = 1$ so that $-1/x = t + C$ and $x(t) = -1/(t+C)$. To find the constant C, we use the initial condition, so that $1 = x(0) = -1/C$ and $C = -1$. Thus, $x(t) = 1/(t-1)$ and $x(t) \to -\infty$ as $t \to 1$ – that is, $x(t)$ blows-up in finite time, and $x(1)$ is not defined.*

As this example demonstrates, even in one-dimension, it is generally not possible to find a solution to nonlinear ODEs which exists for all time $t > 0$; rather, there is usually a small time interval of existence on which we have a differentiable solution.

Consider the system of ODEs

$$\frac{dy}{dt}(t) = F(y(t), t) \text{ for } \; t > 0\,, \tag{3.1a}$$

$$y = y_0 \qquad \text{for } \; t = 0\,. \tag{3.1b}$$

One can think of the variable t as "time", and the solution $t \mapsto y(t)$ is a curve in $\mathbb{R}^n$. Here $y(t) = (y_1(t), y_2(t), ..., y_n(t))$ is a time-dependent n-vector modeling the *state of the system* at time t.

The *evolution equation* (3.1a) defines the so-called dynamical system (or the dynamics) for $t > 0$, while the *initial condition* (3.1b) prescribes the initial state of the system at $t = 0$, in this case, we call that initial state y_0. We are now ready to state the main existence and uniqueness theorem. In the theorem we write

$$\overline{B}(y_0, r) = \{z \in \mathbb{R}^n \mid \|y_0 - z\| \leq r\}\,.$$

Theorem 3.12 (Fundamental theorem of ODEs)**.** *Let $F : \overline{B}(y_0, r) \times [-a, a] \to \mathbb{R}^n$ be a continuous mapping, and let*

$$M = \sup_{y \in \overline{B}(y_0,r), t\in[-a,a]} \|F(y,t)\| .$$

Suppose there is a constant K such that

$$\|F(y_1,t) - F(y_2,t)\| \le K\|y_1 - y_2\| \quad \forall y_1, y_2 \in \overline{B}(y_0,r), t \in [-a,a] . \tag{3.2}$$

Let

$$b < \min\{a, \frac{r}{M}, \frac{1}{K}\} .$$

Then there is a unique continuously differentiable map $y : [-b, b] \to \overline{B}(y_0, r) \subset \mathbb{R}^n$ which is the solution of equation (3.1).

Remark 3.13. *The condition (3.2) is called the Lipschitz condition, and K is called the Lipschitz constant. Whenever F is continuously differentiable on $\overline{B}(y_0, r)$ with bounds that are uniform in t, the Lipschitz condition is satisfied. To see this, let us use $D_yF(y, t)$ to denote the derivative of F with respect to y. Then if*

$$\|D_yF(y,t) \cdot z\| \le K\|z\| \quad \forall\ z \in \mathbb{R}^n\, , t \in [-a,a]\, , y \in \overline{B}(y_0,r)\, ,$$

then (3.2) holds. The chain-rule verifies this claim:

$$\frac{d}{ds}F(z + s(y-z),t) = D_yF(z+s(y-z),t)\cdot(y-z)\, ,$$

so integrating between $s = 0$ and $s = 1$,

$$F(y,t) - F(z,t) = \int_0^1 D_yF(z+s(y-z),t)\cdot(y-z)ds\, .$$

Taking absolute values then yields the result. Thus, whenever F is of class C^1, we are guaranteed to satisfy the conditions of Theorem 3.12.

By the fundamental theorem of calculus, equation (3.1) is equivalent to the integral equation

$$y(t) = y_0 + \int_0^t F(y(s),s)ds\, . \tag{3.3}$$

We will establish existence of the method of successive approximations: Guess $y_1(t)$ such that $y_1(0) = y_0$, for example $y_1(t) = y_0$. Then set

$$y_n(t) = y_0 + \int_0^t F(y_{n-1}(s),s)ds\, .$$

The objective is then to show that as $n \to \infty$, $y_n(t)$ converges to a (unique) solution (3.3) and hence (3.1).

Proof. Consider $C([-b,b];\mathbb{R}^n)$ which is a complete metric space. (Note that $C([-b,b];\mathbb{R}^n) = C_b([-b,b];\mathbb{R}^n)$ since $[-b,b]$ is compact.) Let

$$\mathcal{Y} = \{y \in C([-b,b];\mathbb{R}^n) \mid y(0) = y_0 \text{ and } y(t) \in \overline{B}(y_0, r)\}\,.$$

Then $\mathcal{Y}$ is closed, and since a closed subset of a complete metric space is complete, $\mathcal{Y}$ is a complete metric space. We will apply the contraction mapping theorem to the space $\mathcal{Y}$.

Define the map T by

$$T(y)(t) = y_0 + \int_0^t F(y(s), s)ds \quad t \in (-b, b)\,.$$

We must show that $T(y)$ is in $\mathcal{Y}$ whenever $y \in \mathcal{Y}$. To see that $T(y)$ is a continuous function, let t_n be a sequence such that $t_n \to t$ as $n \to \infty$; we show that $\lim_{n\to\infty} T(y)(t_n) = T(y)(t)$. But

$$\begin{aligned}\lim_{n\to\infty} T(y)(t_n) &= \lim_{n\to\infty} \int_0^{t_n} F(y(s), s)ds \\ &= \lim_{n\to\infty} \int_{-b}^{b} \mathbf{1}_{[0,t_n]}(s) F(y(s), s)ds\,,\end{aligned}$$

where $\mathbf{1}_{[0,t_n]}(s) = 1$ if $s \in [0, t_n]$ and $\mathbf{1}_{[0,t_n]}(s) = 0$ otherwise. Since

$$\mathbf{1}_{[0,t_n]}(s) F(y(s), s) \to \mathbf{1}_{[0,t]}(s) F(y(s), s) \quad \text{uniformly on } [-b, b]\,,$$

then

$$\lim_{n\to\infty} \int_0^{t_n} F(y(s), s)ds = \int_0^t F(y(s), s)ds$$

by Theorem 1.32 so this establishes continuity; hence $T(y) \in C([-b,b];\mathbb{R}^n)$. Furthermore we see that $T(y)(0) = y_0$. Thus, it remains to show that $T(y) \in \overline{B}(y_0, r)$ which is the same as showing that $\|T(y) - y_0\| \le r$. Now, for all $t \in [-b, b]$,

$$\|T(y)(t) - y_0\| = \left\| \int_0^t F(y(s), s)ds \right\| \le \int_0^t \|F(y(s), s)\|ds \le b \cdot M < r\,,$$

since $b < r/M$. This shows that $T(y) \in \mathcal{Y}$ and that $T : \mathcal{Y} \to \mathcal{Y}$.

It remains to show that $T : \mathcal{Y} \to \mathcal{Y}$ is a contraction mapping. For $y_1, y_2 \in \mathcal{Y}$,

$$\begin{aligned}
\sup_{t\in[-b,b]} \|T(y_1)(t) - T(y_2)(t)\| &= \sup_{t\in[-b,b]} \left\| \int_0^t [F(y_1(s), s) - F(y_2(s), s)]\, ds \right\| \\
&\leq \sup_{t\in[-b,b]} \int_0^t \|F(y_1(s), s) - F(y_2(s), s)\| ds \\
&\leq \sup_{t\in[-b,b]} \int_0^t K\|y_1(s) - y_2(s)\| ds \\
&\leq K\, b \sup_{t\in[-b,b]} \|y_1(t) - y_(t)\| \,.
\end{aligned}$$

Since $K\, b < 1$, letting $\lambda = k\, B$, we see that $d(T(y_1), T(y_2)) \leq \lambda d(y_1, y_2)$ so that T is a contraction, and hence there exists a unique fixed-point $y \in \mathcal{Y}$ such that $T(y) = y$. $\square$

Example 3.14. *Let us find b for the ODE in Example 3.11, $dy/dt = y^2$, $y(0) = 1$. At first, we will keep the time-interval $[-a, a]$ in the statement of the theorem unspecified, and we will also keep the radius r unspecified. We compute M:*

$$\begin{aligned}
M &= \sup\{|F(y,t)| \;:\; t \in [-a,a]\,,\; y \in \overline{B}(1,r)\} \\
&= \sup\{y^2 \;:\; y \in \overline{B}(1,r)\} \\
&= (1+r)^2 \,.
\end{aligned}$$

Hence $r/M = r/(1+r)^2$. Also $D_yF(y,t) = \partial F/\partial y(y,t) = 2y$, so

$$K = \sup\{2|y| \;:\; y \in \overline{B}(1,r)\} = 2(1+r)\,.$$

Since F does not explicitly depend on t, a is not involved in the continuity criterion, so we an choose it to be large enough that it does not alter the value of b. We let $a = 1000$. Then, according to the FTODE, we must choose

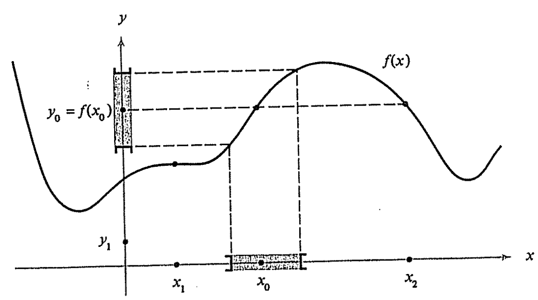
$$b < \min\left\{ \frac{r}{(1+r)^2}\,, \frac{1}{2(1+r)} \right\}\,.$$

The largest value of b occurs for $r = 1$, for which we find that $t \in [0, b]$ for $b < 1/4$. Thus, the theorem does not provide the optimal time of existence given by $t \in [0, 1)$, which we found by directly solving for $y(t) = 1/(1-t)$. On the other hand if we take $t = 1/4$ we see that $y(1/4) = 1/(3/4) = 4/3$, and we can apply the theorem again, this time with $t = 1/4$ playing the role of initial time, and $4/3$ playing the role of initial condition. This allows us to continue the solution for $t < 1$, but we cannot go past $t = 1$.

3.5 Inverse function theorem

By definition, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable at x and $\det Df(x) \neq 0$, then $Df(x)$ is a linear isomorphism, meaning that $Df(x)$ is an invertible matrix. The geometric picture here, is that $f : \mathbb{R}^n \to \mathbb{R}^n$ is some nonlinear function, and at the point $x \in \mathbb{R}^n$, the best linear approximation to f, given by $Df(x)$, is invertible, so perhaps (at least locally near x) the nonlinear function f is also invertible.

To gain intuition, consider a function $f : \mathbb{R} \to \mathbb{R}$, draw the graph($f$) by plotting the points $y = f(x)$, and then rotate the graph by 90^o. This rotated picture, now viewed as a function over the y-axis, is a graph only if over each point y the curve has only one value. If f is of class C^1, and if $f'(x_0) \neq 0$, then f is invertible (one-to-one) in a small enough neighborhood of x_0. Geometrically, this is very clear, since $f'(x_0) \neq 0$, means that f has a non-zero slope at x_0 and hence a nonzero slope nearby x_0.



We will focus on *local invertibility* of the function f, meaning the invertibility of $f(x)$ for x near x_0 and for y near $y_0 = f(x_0)$. Suppose that the inverse function exists so that $x = f^{-1}(y)$. In this case, we can employ the chain-rule to compute the derivative of f^{-1}. Observe that

$$f^{-1}(f(x)) = x\,.$$

Then

$$\frac{d(f^{-1}\circ f)}{dx}(x) = \frac{df^{-1}}{dy}(f(x))\,\frac{\partial f}{\partial x}(x) = 1\,,$$

so that

$$\left.\frac{df^{-1}}{dy}\right|_{y=f(x)} = \frac{1}{df/dx}\,.$$

To actually verify that f^{-1} is differentiable requires a good deal more care.

If $f'(x_0) = 0$, then f may or may not be invertible near x_0; in the above figure, f is not invertible near x_1, but $f(x) = x^3$ is invertible near $x_0 = 0$. In the case that $f'(x_0) = 0$, further analysis is required to determine invertibility. Furthermore, the condition $f'(x_0) \neq 0$ does not guarantee that we can solve $f(x) = y$ for all y. For example, looking at the above figure, we see that there does not exist any point x such that $f(x) = y_1$. Even when solutions to exist, they need not be unique. The above figure shows that $f(x_0) = f(x_2)$. Thus, we can only expect to have a unique solution in a sufficiently small neighborhood of x_0

In general, f is only invertible near $f(x_0)$, that is, only for y close to $f(x_0)$ can we uniquely find some x near x_0 such that $f(x) = y$.

Theorem 3.15 (Inverse Function Theorem)**.** *Let $A \subset \mathbb{R}^n$ denote an open set, and let $f \in C^1(A; \mathbb{R}^n)$. Suppose that for $x_0 \in A$, $\det Df(x_0) \neq 0$. Then there exists a neighborhood U of x_0 in A and an open neighborhood W of $f(x_0)$ such that $f(U) = W$, and f has a C^1 inverse $f^{-1} : W \to U$.*

Moreover, for $y \in W$, $x = f^{-1}(y)$, we have

$$Df^{-1}(y) = [Df(x)]^{-1},$$

the inverse of $Df(x)$ meaning the inverse of the linear mapping (corresponding to inverting the matrix).

Whenever the inverse function f^{-1} of f exists, we can uniquely solve the nonlinear equation $f(x) = y \in W$ with some $x \in U$.

Example 3.16. *Consider the equations $(x^4 + y^4)/x = u(x, y)$, $\sin x + \cos y = v(x, y)$. Near which points (x, y) can we solve for (x, y) in terms of (u, v)?*

Here the nonlinear functions are given by $f_1(x, y) = (x^4 + y^4)/x$ and $f_2(x, y) = \sin x + \cos y$ and we have

$$\begin{aligned} f_1(x, y) &= u \\ f_2(x, y) &= v\,. \end{aligned}$$

The domain of the function $f = (f_1, f_2)$ can be taken to be $A = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$. We compute the 2×2 matrix $Df(x, y)$:

$$Df(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{3x^4 - y^4}{x^2} & \frac{4y^3}{x} \\ \cos x & -\sin y \end{pmatrix},$$

so that

$$\det Df(x, y) = \frac{\sin y}{x^2}(y^4 - 3x^4) - \frac{4y^3}{x}\cos x\,.$$

According to the inverse function theorem, we can solve for (x, y) in a neighborhood of points $(x_0, y_0) \in A$ where $\det Df(x_0, y_0) \neq 0$. Thus, we are looking for points (x_0, y_0) such that $x_0 \neq 0$ and $\sin y(y^4 - 3x^4) \neq 4xy^3 \cos x$. In general, one cannot explicitly solve for all such points, but, for example, $(x_0, y_0) = (\pi/2, \pi/2)$ does the job.

Example 3.17. *Let $f = (f_1, f_2)$ with $f_1(x, y) = e^x \cos y$ and $f_2(x, y) = e^x \sin y$. Then f is invertible locally, but not globally.*

First, we compute

$$Df(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$$

so that

$$\det Df(x, y) = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0 \,.$$

Then, using the inverse function theorem, we see that f is locally invertible. On the other hand, f cannot be globally invertible on $\mathbb{R}^2$ since f is not a one-to-one map; in particular,

$$f_1(x, y + 2\pi) = f_1(x, y) \,, \quad f_2(x, y + 2\pi) = f_2(x, y) \,.$$

Note that for functions $f : \mathbb{R} \to \mathbb{R}$, if f is differentiable and if $f'(x) \neq 0$ for all $x \in \mathbb{R}$, then $f'(x)$ is either strictly positive or strictly negative (since $f'(x)$ satisfies the intermediate value theorem; hence, f must be globally one-to-one as f is always increasing or decreasing. An example is $f = \tanh : \mathbb{R} \to \mathbb{R}$ is one such example. Example 3.17 shows that this need not be the case in $\mathbb{R}^2$.

Before proceeding with the proof of the inverse function theorem, we will need an important technical lemma.

Definition 3.18. *Let $L(\mathbb{R}^n; \mathbb{R}^n)$ denote the set of all $n \times n$ matrices (or linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, and let $GL(\mathbb{R}^n; \mathbb{R}^n)$ denote the subset of all invertible $n \times n$ matrices (or invertible linear maps of $\mathbb{R}^n$ to $\mathbb{R}^n$). Thus,*

$$GL(\mathbb{R}^n; \mathbb{R}^n) = \{B \in L(\mathbb{R}^n; \mathbb{R}^n) \mid \det B \neq 0\} \,.$$

The space $GL(\mathbb{R}^n; \mathbb{R}^n)$ is called the general linear group.

Definition 3.19. *Let $\mathrm{Inv} : GL(\mathbb{R}^n; \mathbb{R}^n) \to GL(\mathbb{R}^n; \mathbb{R}^n)$ be the map that takes $B \in GL(\mathbb{R}^n; \mathbb{R}^n)$ to its inverse B^{-1}, so that $\mathrm{Inv}(B) = B^{-1}$.*

Lemma 3.20.

(i) $GL(\mathbb{R}^n; \mathbb{R}^n)$ is an open subset of $L(\mathbb{R}^n; \mathbb{R}^n)$;

(ii) $\mathrm{Inv} : GL(\mathbb{R}^n; \mathbb{R}^n) \to GL(\mathbb{R}^n; \mathbb{R}^n)$ is a C^∞ mapping.

Proof. (i) The determinant mapping $\det : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$ is an n-linear map (recall that the determinant is linear in the rows of the matrix). Since multilinear maps are continuous, the determinant map is continuous. Furthermore, just as we showed in Example 2.15, that the derivative of a linear map is equal to the linear map, so too is the derivative of a multilinear map the multilinear map itself, so that the determinant map is also differentiable. Because the set consisting of the singleton $\{0\}$ is closed, we have that $\det^{-1}(\{0\})$ is closed. Hence $L(\mathbb{R}^n;\mathbb{R}^n)\backslash\{0\}$ is open, and this open set is exactly equal to $GL(\mathbb{R}^n;\mathbb{R}^n)$.

(ii) Given $B \in GL(\mathbb{R}^n;\mathbb{R}^n)$, the inverse matrix $B^{-1} = (\det B)^{-1}[\mathrm{Cof}(B)]^T$, where $\mathrm{Cof}(B)]^T$ denotes the transpose of the matrix of cofactors of the matrix B, where

$$\mathrm{Cof}(B)_{ij} = (-1)^{i+j} M_{ij}\,,$$

where M_{ij} is the determinant of the minor matrix formed by removing the ith row and jth column from B. As the function $B \mapsto (\det B)^{-1}$ is differentiable, it suffices to show that the mapping $\mathrm{Cof} : L(\mathbb{R}^n;\mathbb{R}^n) \to L(\mathbb{R}^n;\mathbb{R}^n)$, which takes a matrix to its cofactor, is C^∞. Since each component of the cofactor matrix is just the determinant mapping, and as the determinant mapping is C^∞, being a multi-linear map, we see that Cof is C^∞ as well. □

Proof of Theorem 3.15.

Step 1. Simplification to the case that $Df(x_0) = \mathrm{Id}$. In this step, we show that we can reduce the analysis to the special case that $Df(x_0)$ is the identity matrix, Id.

Let $T = Df(x_0)$; then T^{-1} exists and by the chain-rule,

$$D(T^{-1}\circ f)(x_0) = DT^{-1}(f(x_0))\, Df(x_0) = T^{-1}\, Df(x_0) = \mathrm{Id}\ .$$

Now if the theorem is true for $T^{-1}\circ f$, then the theorem is also true for f, for if g is an inverse of for $T^{-1}\circ f$, then the inverse for f is $g\circ T^{-1}$.

It is convenient to make one further simplification; namely, we can assume that $x_0 = 0$ and $f(x_0) = 0$. Suppose that we have established the theorem for this particular case that $x_0 = 0$ and $f(x_0) = 0$; to see how the general case is obtained from this special case, let $h(x) = f(x + x_0) - f(x_0)$. Then $h(0) = 0$ and $Dh(0) = Df(x_0)$, so $Dh(0)$ is invertible. Then if h has an inverse near $x = 0$, the required inverse for f near x_0 is given by

$$f^{-1}(y) = h^{-1}(y - f(x_0)) + x_0\,.$$

Thus we have shown that it suffices to consider $x_0 = 0$, $f(x_0) = 0$, and $Df(x_0) = \mathrm{Id}$.

Step 2. Local inverse for f via the contraction mapping theorem. With the simplification of Step 1, we wish to find an open neighborhood $\mathcal{Y}$ of $f(x_0) = 0$ and an open neighborhood $\mathcal{X}$ of $x_0 = 0$ such that given any $y \in \mathcal{Y}$, there exists a unique $x \in \mathcal{X}$ with $f(x) = y$. In order to do so, we consider the function

$$g_y(x) = y + x - f(x)\,, \quad (y \text{ is a parameter for the function } g)\,.$$

The idea is to show that g is a contraction mapping, in which case there exists a unique fixed-point $x \in \mathcal{X}$ such that $g_y(x) = x$ which is the same as $x = y + x - f(x)$ which is the same as $y = f(x)$, which is desired.

We begin by first defining the function $g(x)$ as

$$g(x) = x - f(x)\,, \tag{3.4}$$

which implies that $g(0) = 0$ and that

$$Dg(0) = 0\,.$$

Since f is assumed to be C^1 in a neighborhood of $x_0 = 0$, then g is also C^1 on this neighborhood, and hence Dg is continuous on this neighborhood; therefore, for $r > 0$ small and with $\|x\| < r$, $\|Dg_i(x)\| < 1/(2\sqrt{n})$ where $g = (g_1, ..., g_n)$. By the mean value theorem, Theorem 2.38, given $x \in B(0,r) = \{x \in \mathbb{R}^n \ : \ \|x\| < r\}$, there are points $c_1, c_2, ..., c_n$ such that

$$g_i(x) = g_i(x) - g_i(0) = Dg_i(c_i) \cdot (x - 0) = Dg_i(c_i) \cdot x \qquad i = 1, ..., n\,.$$

We have that

$$\|g(x)\|^2 = \sum_{i=1}^n |g_i(x)|^2 = \sum_{i=1}^n |Dg_i(c_i) \cdot x|^2 \le \sum_{i=1}^n \|Dg_i(c_i)\|^2 \, \|x\|^2 < \frac{\|x\|^2}{4} < \frac{r^2}{4}\,,$$

so that $\|g(x)\| < r/2$. Hence, we have shown that

$$g : \overline{B}(0,r) \to \overline{B}(0,r/2)\,.$$

Next, let $y \in \overline{B}(0,r/2)$; then, $g_y : \overline{B}(0,r) \to \overline{B}(0,r)$. To see this, note that for $\|y\| \le r/2$ and $x \in \overline{B}(0,r)$, we have that

$$\|g_y(x)\| = \|y + g(x)\| \le \|y\| + \|g(x)\| < \frac{r}{2} + \frac{r}{2} = r\,.$$

Now let x_1 and x_2 be any two points in $\overline{B}(0,r)$. Then $\|g_y(x_1) - g_y(x_2)\| = \|g(x_1) - g(x_2)\|$ and by the mean value theorem,

$$\|g(x_1) - g(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|\,.$$

This shows that g_y is a contraction on the complete metric space $\overline{B}(0,r)$, and hence that $f(x) = y$ for any $y \in \overline{B}(0,r/2)$. This means that f has an inverse $f^{-1} : \overline{B}(0,r/2) \subset \mathbb{R}^n \to \overline{B}(0,r) \subset \mathbb{R}^n$.

Step 3. f^{-1} is continuous. Let $x_1, x_2 \in \overline{B}(0,r)$. From (3.4), we see that

$$\|x_1 - x_2\| \le \|f(x_1) - f(x_2)\| + \|g(x_1) - g(x_2)\| \le \|f(x_1) - f(x_2)\| + \frac{1}{2}\|x_1 - x_2\|\,,$$

so that $\|x_1 - x_2\| \le 2\|f(x_1) - f(x_2)\|$. It follows that if $y_1, y_2 \in \overline{B}(0, r/2)$, then

$$\|f^{-1}(y_1) - f^{-1}(y_2)\| \le 2\|y_1 - y_2\|\,,$$

and hence f^{-1} is continuous.

Step 4. f^{-1} **is differentiable on** $B(0, r/2)$ **for** $r > 0$ **sufficiently small.** By assumption $Df(0)$ is invertible, $Df : A \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuous, and by Lemma 3.20, $GL(\mathbb{R}^n; \mathbb{R}^n)$ is open in $L(\mathbb{R}^n; \mathbb{R}^n)$. Thus, for x in a sufficiently small neighborhood about 0, $[Df(x)]^{-1}$ exists If this neighborhood does not contain $\overline{B}(0, r/2)$, then we take $r > 0$ even smaller. Hence, we can assume that that $[Df(x)]^{-1}$ exists for all $x \in B(0, r/2)$. Moreover, since Df is continuous and since inversion is a smooth operator according to Lemma 3.20, we see that $[Df(x)]^{-1}$ is continuous on $\overline{B}(0, r/2)$ and hence uniformly continuous so that

$$\|[Df(x)]^{-1} \cdot y\| \le M\|y\| \quad \forall \;\; x \in \overline{B}(0, r/2), y \in \mathbb{R}^n\,.$$

Now for $y_1, y_2 \in B(0, r/2)$, $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$,

$$\begin{aligned}
&\frac{\|f^{-1}(y_1) - f^{-1}(y_2) - [Df(x_2)]^{-1} \cdot (y_1 - y_2)\|}{\|y_1 - y_2\|} \\
&\qquad = \frac{\|x_1 - x_2 - [Df(x_2)]^{-1} \cdot (f(x_1) - f(x_2))\|}{\|f(x_1) - f(x_2)\|} \\
&\qquad = \left[\frac{\|x_1 - x_2}{\|f(x_1) - f(x_2)\|}\right] \frac{\|[Df(x_2)]^{-1} \cdot \{Df(x_2) \cdot (x_1 - x_2) - (f(x_1) - f(x_2))\}\,\|}{\|f(x_1) - f(x_2)\|}\,.
\end{aligned}$$

Since $\|x_1 - x_2\| \le 2\|f(x_1) - f(x_2)\|$ together with $\|[Df(x)]^{-1} \cdot y\| \le M\|y\|$ shows that

$$\begin{aligned}
&\frac{\|f^{-1}(y_1) - f^{-1}(y_2) - [Df(x_2)]^{-1} \cdot (y_1 - y_2)\|}{\|y_1 - y_2\|} \\
&\qquad\qquad \le 2M \frac{\|Df(x_2) \cdot (x_1 - x_2) - (f(x_1) - f(x_2))\|}{\|x_2 - x_2\|}\,,
\end{aligned}$$

which converges as $\|x_1 - x_2\| \to 0$ by the assumed differentiability of f at x_2. This shows that f^{-1} is differentiable at y_2 with derivative

$$[Df(x_2)]^{-1} = [Df(f^{-1}(y_2))]^{-1}\,.$$

To complete the proof, we set $W = B(0, r/2)$ and $U = f^{-1}(W)$. □

3.6 Exercises

Problem 3.1. *(a) Let $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$. Show that the map $(x, y) \mapsto (u, v)$ is locally invertible at all points $(x, y) \neq (0, 0)$.*

(b) Compute $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, $\frac{\partial y}{\partial v}$

Problem 3.2. *Let* $f(x) = x + 2x^2 \sin(1/x)$ *for* $x \neq 0$ *and* $f(0) = 0$*. Show that* $f'(0) \neq 0$*, but that* f *is not locally invertible near* $x = 0$*. Why does this not contradict Theorem 3.15?*

Problem 3.3. *Let* $L : \mathbb{R}^n \to \mathbb{R}^n$ *denote a linear isomorphism (invertible linear transformation), and let* $f(x) = L(x) + g(x)$*, where* $\|g(x)\| \leq M\|x\|^2$ *and* $f \in C^1(\mathbb{R}^n; \mathbb{R}^n)$*. Show that* f *is locally invertible near* $x = 0$*.*

Problem 3.4. *Investigate whether the system*

$$\begin{aligned} u(x, y, z) &= x + xyz \\ v(x, y, z) &= y + xy \\ w(x, y, z) &= z + 2x + 3z^2 \end{aligned}$$

can be solved for (x, y, z) *in terms of* (u, v, w) *near* $(x, y, z) = (0, 0, 0)$*.*

Problem 3.5. *Solve* $\frac{dy}{dt} = 1 + y^2$, $y(0) = 0$ *by the method of successive approximations. Is* $y(t)$ *defined for all* $t \geq 0$*? Compute the* b *in Theorem 3.12 for this equation.*

Problem 3.6. *Show that* $\frac{dy}{dt} = \sqrt{y}$, $y(0) = 0$ *has two solutions:*

$$y(t) = 0 \text{ and } y(t) = \begin{cases} 0, & t \leq 0 \\ \frac{t^2}{4}, & t > 0 \end{cases}.$$

Does this contradict Theorem 3.12?

Problem 3.7. *Consider the equation* $\frac{dy}{dt} = te^{y^2} \sin y$, $y(0) = 1$*. Obtain an estimate on the time interval for which* $y(t)$ *is defined.*

Problem 3.8. *Let* B *denote an* $n \times n$ *matrix and consider the linear system*

$$\frac{dy}{dt} = B \cdot y(t), \quad y(t) \in \mathbb{R}^n$$

Show that a solution is

$$y(t) = e^{tB} y(0)\,, \text{ where } e^B = \sum_{n=0}^{\infty} \frac{B^n}{n!}\,.$$

The solution exists for all $t \geq 0$*; can this fact be derived from Theorem 3.12?*