Math 125B: Winter 2013

Solution to Problem 9.6.11

As illustrated in Figure 1, we choose two constant vectors $a, b \in \mathbb{R}^3$ that are linearly independent from $\gamma'(t_0)$, which is possible since $\gamma'(t_0) \neq 0$. (For example, we can use the normal and binormal vectors to the curve γ at t_0.) We then define $F : I \times \mathbb{R}^2 \to \mathbb{R}^3$ by

$$F(t, u, v) = \gamma(t) + ua + vb.$$

That is, $F(t, u, v)$ is the point in $\mathbb{R}^3$ that is obtained by going u in the a-direction and v in the b-direction from $\gamma(t)$. In particular, $F(t, 0, 0) = \gamma(t)$.

The function F is C^1 since γ is C^1 and

$$\frac{\partial F}{\partial t}(t, u, v) = \gamma'(t), \qquad \frac{\partial F}{\partial u}(t, u, v) = a, \qquad \frac{\partial F}{\partial v}(t, u, v) = b.$$

Moreover, the differential matrix of F is

$$[dF(t, u, v)] = \left[\begin{array}{ccc} \gamma'(t) & a & b \end{array}\right],$$

where $\gamma'(t)$, a, b are interpreted as column vectors. The differential $dF(t_0, 0, 0)$ has rank 3 and is invertible, since $\{\gamma'(t_0), a, b\}$ are linearly independent.

The inverse function theorem implies that there exist neighborhoods U of $(t_0, 0, 0)$ and V of $\gamma(t_0)$ such that $F : U \to V$ has a C^1-inverse $F^{-1} : V \to U$. We write $F^{-1} = (e, f, g)$ where $e, f, g : V \to \mathbb{R}$ are C^1 and

$$t = e(x, y, z), \qquad u = f(x, y, z), \qquad v = g(x, y, z).$$

If $(x, y, z) \in V$, then $(x, y, z) = \gamma(t)$ if and only if $F^{-1}(x, y, z) = (t, 0, 0)$. Thus, in V, the curve γ is the solution of the equations

$$f(x, y, z) = 0, \qquad g(x, y, z) = 0.$$

Geometrically, the curve is the intersection of the two surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$.

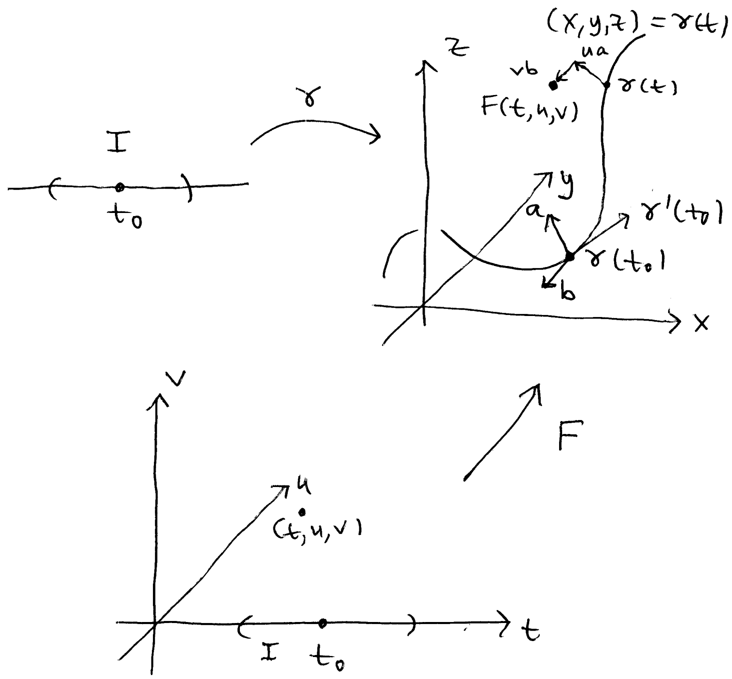

Figure 1: The curve γ and the map F.