Real Analysis
Math 125B, Spring 2013
Solutions: Final

1. Prove that
$$\frac{1}{2} \leq \int_0^1 \frac{2x}{\sqrt{x^{2013} + 2x + 1}}\, dx \leq 1.$$

Solution.

- For $0 \leq x \leq 1$, we have
$$1 \leq \sqrt{x^{2013} + 2x + 1} \leq 2,$$
so
$$x \leq \frac{2x}{\sqrt{x^{2013} + 2x + 1}} \leq 2x.$$
By the monotonicity of the integral,
$$\frac{1}{2} = \int_0^1 x\, dx \leq \int_0^1 \frac{2x}{\sqrt{x^{2013} + 2x + 1}}\, dx \leq \int_0^1 2x\, dx = 1.$$

2. Prove or disprove: if E is a subset of $\mathbb{R}^2$, then the closure of the interior of E is necessarily the same as the closure of E.

Solution.

- This statement is false.
- For example, if $E = \{0\}$ consists of a single point, then $E^\circ = \emptyset$ and $\overline{E^\circ} = \emptyset$, but $\overline{E} = \{0\}$.
- Or, for another example, if $E = \mathbb{Q}^2$, then $E^\circ = \emptyset$ and $\overline{E^\circ} = \emptyset$, but $\overline{E} = \mathbb{R}^2$.

3. Evaluate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2 + k^2}.$$

Solution.

- Consider the integral
$$\int_0^1 f(x)\,dx, \qquad f(x) = \frac{x}{1+x^2}.$$
The function f is integrable on $[0,1]$ since it's continuous (or, alternatively, since it's monotone).

- The function f is increasing on $[0,1]$ since
$$f'(x) = \frac{1-x^2}{(1+x^2)^2} \geq 0.$$
Therefore, in an upper Riemann sum $U(f;P)$ we evaluate f at the right endpoints and in a lower Riemann sum $L(f;P)$, we evaluate f at the left endpoints.

- Let P be the partition of $[0,1]$ into n intervals of equal length $1/n$ with endpoints $x_k = k/n$, where $k = 0, 1, \ldots, n$.

- For this partition,
$$U(f;P) = \frac{1}{n} \sum_{k=1}^{n} \frac{k/n}{1+(k/n)^2} = \sum_{k=1}^{n} \frac{k}{n^2+k^2},$$
$$L(f;P) = \frac{1}{n} \sum_{k=1}^{n} \frac{(k-1)/n}{1+((k-1)/n)^2} = \sum_{k=1}^{n} \frac{k}{n^2+k^2} - \frac{1}{2n}.$$

- Since $L(f;P) \leq \int_0^1 f \leq U(f;P)$, it follows that
$$\int_0^1 \frac{x}{1+x^2}\,dx \leq \sum_{k=1}^{n} \frac{k}{n^2+k^2} \leq \int_0^1 \frac{x}{1+x^2}\,dx + \frac{1}{2n},$$
and the "squeeze" theorem implies that
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2} = \int_0^1 \frac{x}{1+x^2}\,dx = \left[\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1 = \frac{1}{2}\ln 2.$$

4. Suppose that $f : [0, \pi] \to \mathbb{R}$ is a continuously differentiable function. Prove that

$$\lim_{n\to\infty} \int_0^\pi f(x) \sin(nx)\, dx = 0.$$

HINT. Integrate by parts.

Solution.

- Since both f and $\sin nx$ are continuously differentiable on $[0, \pi]$, the integration by parts formula applies, and

$$\begin{aligned}\int_0^\pi f(x)\sin(nx)\, dx &= \left[\frac{-\cos(nx)}{n} \cdot f(x)\right]_0^\pi + \frac{1}{n}\int_0^\pi f'(x)\cos(nx)\, dx \\ &= \frac{f(0) + (-1)^{n+1} f(\pi)}{n} + \frac{1}{n}\int_0^\pi f'(x)\cos(nx)\, dx.\end{aligned}$$

- The limit as $n \to \infty$ of the constant term proportional to $1/n$ is zero.

- For the integral term, either observe that

$$\frac{1}{n} f'(x)\cos(nx) \to 0 \qquad \text{uniformly on } [0, 1]$$

since f' is continuous and therefore bounded, and use the fact that we can exchange the order of uniform limits and integration, or estimate the integral directly:

$$\begin{aligned}\left|\frac{1}{n}\int_0^\pi f'(x)\cos(nx)\, dx\right| &\leq \frac{1}{n}\int_0^1 |f'(x)|\, dx \\ &\leq \frac{1}{n}\sup_{[0,1]} |f'| \to 0 \qquad \text{as } n \to \infty.\end{aligned}$$

Remark. This result says that the Fourier sine coefficients

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\, dx$$

of a continuously differentiable function f approach zero as $n \to \infty$. It's a special case of the Riemann-Lebesgue lemma. The general result for a Lebesgue integrable function f such that

$$\int_0^\pi |f(x)|\, dx < \infty$$

follows by approximating f with smooth functions and using the proof above.

5. Let $p > 0$. Define the following improper Riemann integral as a limit of Riemann integrals:

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx.$$

For what values of p does this integral converge? Hint. Use the substitution $u = \ln x$.

Solution.

- The improper integral is defined by

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^p}\,dx.$$

  The Riemann integral on the right-hand side exists since the integrand is continuous on $[2, b]$.

- The substitution formula with $u = \ln x$ and $du = dx/x$ gives

$$\begin{aligned} \int_2^b \frac{1}{x(\ln x)^p}\,dx &= \int_{\ln 2}^{\ln b} \frac{1}{u^p}\,du. \\ &= \left[\frac{u^{1-p}}{1-p}\right]_{\ln 2}^{\ln b} \\ &= \frac{(\ln b)^{1-p} - (\ln 2)^{1-p}}{1-p} \end{aligned}$$

  where we assume that $p \neq 1$.

- Since $\ln b \to \infty$ as $b \to \infty$, this integral diverges if $0 < p < 1$, and converges if $p > 1$ to

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \frac{1}{(p-1)(\ln 2)^{p-1}}.$$

- The integral also diverges (very, very slowly) if $p = 1$ since

$$\int_2^b \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\ln b} \frac{1}{u}\,du = \ln\ln b - \ln\ln 2 \to \infty.$$

6. Suppose that $f : [a, b] \to \mathbb{R}$ is a nonzero, Riemann integrable function such that $1/f : [a, b] \to \mathbb{R}$ is bounded. Prove that $1/f$ is Riemann integrable.

Solution.

- Suppose that

$$\frac{1}{|f(x)|} \le M \qquad \text{for } a \le x \le b.$$

- Then for all $x, y \in [a, b]$, we have

$$\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| = \frac{|f(x) - f(y)|}{|f(x)f(y)|} \le M^2 |f(x) - f(y)|.$$

  This inequality implies that (see Proposition 2.19 in the class notes on sups and infs) for every subset $I \subset [a, b]$ we have

$$\sup_I \frac{1}{f} - \inf_I \frac{1}{f} \le M^2 \left(\sup_I f - \inf_I f \right).$$

- For every partition $P = \{I_1, I_2, \dots, I_n\}$ of $[a, b]$, we have

$$\begin{aligned} U\left(\frac{1}{f}; P\right) - L\left(\frac{1}{f}; P\right) &= \sum_{k=1}^{n} \left(\sup_{I_k} \frac{1}{f} - \inf_{I_k} \frac{1}{f} \right) |I_k| \\ &\le M^2 \sum_{k=1}^{n} \left(\sup_{I_k} f - \inf_{I_k} f \right) |I_k|. \end{aligned}$$

  Therefore, $1/f$ satisfies the Cauchy criterion for integrability if f does, and it follows that $1/f$ is integrable if f is nonzero and integrable and $1/f$ is bounded.

7. Define $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f_1(x_1, x_2) = e^{x_1} \cos x_2, \qquad f_2(x_1, x_2) = e^{x_1} \sin x_2.$$

(a) Why is f differentiable on $\mathbb{R}^2$? Compute the differential matrix of f.
(b) Evaluate the directional derivative $D_{(3/5,4/5)}f(0, \pi/2)$ of f at $P = (0, \pi/2)$ in the direction $e = (3/5, 4/5)$. Which component f_1, f_2 is increasing at P in the direction e?
(c) What does the implicit function theorem say about the existence of local inverses of f? Does f has a global inverse $f^{-1} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$?

Solution.

- (a) The functions f_1, f_2 have continuous partial derivatives on $\mathbb{R}^2$, so f is differentiable on $\mathbb{R}^2$. The differential matrix is

$$[df] = \begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix} = \begin{pmatrix} e^{x_1} \cos x_2 & -e^{x_1} \sin x_2 \\ e^{x_1} \sin x_2 & e^{x_1} \cos x_2 \end{pmatrix}.$$

- (b) The differential matrix of f at $(0, \pi/2)$ is

$$[df(0, \pi/2)] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

  so the directional derivative is

$$D_{(3/5,4/5)}f(0, \pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}.$$

  The component f_2 is increasing in the direction e since its directional derivative is positive.

- (c) The derivative of f has determinant

$$\det df = e^{x_1} \cos^2 x_2 + e^{x_1} \sin^2 x_2 = e^{x_1} > 0$$

  so $df(x_1, x_2)$ is invertible at every $(x_1, x_2) \in \mathbb{R}^2$, and f is C^1. The inverse function theorem implies that there are open neighborhoods U of (x_1, x_2) and V of $(f_1(x_1, x_2), f_2(x_1, x_2))$ such that $f : U \to V$ is one-to-one and onto with C^1-inverse $f^{-1} : V \subset \mathbb{R}^2 \to U \subset \mathbb{R}^2$.

- Although f is locally invertible at every point is is not globally invertible since it is not one-to-one:

$$f(x_1, x_2 + 2n\pi) = f(x_1, x_2)$$

for every $n \in \mathbb{Z}$.

Remark. This function correspond to the complex exponential function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = e^z$. The local inverse is a branch of the complex logarithm $f^{-1}(z) = \log z$, but the logarithm can't be extended to a single-valued, differentiable function on $\mathbb{C}$.

8. Suppose that $0 < a < b$ and $0 < \delta < \pi/2$. Let A be the region

$$A = \left\{(x, y) \in \mathbb{R}^2 : a^2 \leq x^2 + y^2 \leq b^2 \text{ and } 0 \leq \tan^{-1}(y/x) \leq 2\pi - \delta\right\},$$

and consider the integral

$$I = \int_A e^{-(x^2+y^2)}\, dxdy.$$

Make the change of coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta$$

in this integral and evaluate it. Justify your steps.

Solution.

- The change of coordinates is a C^1-diffeomorphism in an open neighborhood U of A that doesn't contain the origin e.g.,

$$U = \left\{(x, y) \in \mathbb{R}^2 : a^2 - \epsilon < x^2 + y^2 < b^2 + \epsilon, \right. \\ \left. \text{and } -\epsilon < \tan^{-1}(y/x) < 2\pi - \delta + \epsilon\right\}$$

  for sufficiently small $\epsilon > 0$, so we can apply the change of variables theorem.

- The Jacobian determinant is

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

  so formally $dxdy = rdrd\theta$ and

$$\int_A e^{-(x^2+y^2)}\, dxdy = \int_R e^{-r^2}\, rdrd\theta$$

  where A is the image of the rectangle

$$R = \{(r, \theta) : a \leq r \leq b, \quad 0 \leq \theta \leq 2\pi - \delta\}.$$

- Since it's continuous, the function re^{-r^2} is integrable on the rectangle, as are its iterated integrals. Fubini's theorem implies that

$$\int_R e^{-r^2}\, rdrd\theta = \int_0^{2\pi-\delta} \left(\int_a^b re^{-r^2}\, dr\right) d\theta.$$

- Making the substitution $u = r^2$, $du = 2r\,dr$, we get

$$\int_a^b re^{-r^2}\,dr = \frac{1}{2}\int_{a^2}^{b^2} e^{-u}\,du = \frac{1}{2}\left[-e^{-u}\right]_{a^2}^{b^2} = \frac{1}{2}\left(e^{-a^2} - e^{-b^2}\right).$$

  Thus,

$$\begin{aligned}\int_A e^{-(x^2+y^2)}\,dxdy &= \frac{1}{2}\int_0^{2\pi-\delta}\left(e^{-a^2} - e^{-b^2}\right)\,d\theta \\ &= \frac{1}{2}(2\pi - \delta)\left(e^{-a^2} - e^{-b^2}\right).\end{aligned}$$

Remark. By considering the improper integral with $a \to 0^+$, $b \to \infty$, and $\delta \to 0^+$, we get that

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)} = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr = \pi.$$

This is the classic trick (apparently due to Laplace) for evaluating the definite Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$