Selected Homework Solutions
Math 125B: Winter 2013

9.1.8

- At $y = 0$, we have

$$f(x, 0) = \begin{cases} x^{2-2p} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

  Thus,

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h}{|h|^{2p}}.$$

  It follows that $\partial f / \partial x(0, 0)$ doesn't exist if $p \geq 1/2$, and is 0 if $p < 1/2$.

- By the quotient and chain rules, the partial derivative of f with respect to x exists if $(x, y) \neq (0, 0)$ and is given by

$$\frac{\partial f}{\partial x} = \frac{2x}{(x^2 + y^2)^p} - \frac{2px^3}{(x^2 + y^2)^{p+1}}.$$

  We have

$$\left| \frac{x}{(x^2 + y^2)^p} \right|, \left| \frac{x^3}{(x^2 + y^2)^{p+1}} \right| \leq \frac{1}{(x^2 + y^2)^{p-1/2}},$$

  so

$$\frac{\partial f}{\partial x}(x, y) \to 0 \qquad \text{as } (x, y) \to (0, 0)$$

  if $p < 1/2$, and in that case $\partial f / \partial x$ is continuous at $(0, 0)$.

- Thus, $\partial f / \partial x$ is continuous at $(0, 0)$ whenever it exists at $(0, 0)$.

9.2.4

- The differential matrix of G at (x, y) is

$$[dG(x,y)] = \begin{pmatrix} y/x & \ln x \\ e^y & xe^y \\ y\cos xy & x\cos xy \end{pmatrix}.$$

- The differential matrix of G at $(1, \pi)$ is

$$\begin{pmatrix} \pi & 0 \\ e^\pi & e^\pi \\ -\pi & -1 \end{pmatrix}.$$

- The best affine approximation of G at $(1, \pi)$ is given by

$$\begin{aligned} T(x,y) &= (0, e^\pi, 0) \\ &\quad + (\pi(x-1), e^\pi(x-1+y-\pi), -\pi(x-1)-(y-\pi)). \end{aligned}$$

9.2.10

- The function has first-order partial derivatives at every point of $\mathbb{R}^2$.
- For $(x, y) \neq (0, 0)$ the partial derivatives exist by the quotient rule.
- For $(x, y) = (0, 0)$, the partial derivatives exist since $f(x, 0) = x$ and $f(0, y) = 0$ for all $x, y \in \mathbb{R}$, so

$$\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx} f(x,0)\bigg|_{x=0} = 1, \quad \frac{\partial f}{\partial y}(0,0) = \frac{d}{dy} f(0,y)\bigg|_{y=0} = 0.$$

- The function is not differentiable at $(0, 0)$, even though it is continuous at $(0, 0)$.
- To prove this, note that if f was differentiable at $(0, 0)$ with derivative A then its differential matrix would be

$$[A] = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0) \right) = (1, 0),$$

  and the best linear approximation would be

$$f(0, 0) + A(x, y) = x.$$

- It follows that f is differentiable at $(0, 0)$ if and only if the limit that defines the derivative,

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - A(x,y)}{\|(x,y)\|} &= \lim_{(x,y)\to(0,0)} \frac{x^3/(x^2+y^2) - x}{(x^2+y^2)^{1/2}} \\ &= \lim_{(x,y)\to(0,0)} \frac{-xy^2}{(x^2+y^2)^{3/2}}, \end{aligned}$$

  exists and is equal to 0.

- However, this limit doesn't exist. For example, if we set $x = r\cos\theta$, $y = r\sin\theta$, then

$$\frac{-xy^2}{(x^2+y^2)^{3/2}} = -\cos\theta \sin^2\theta,$$

  and we get different limits as $r \to 0$ in the directions $\theta = 0$ (the limit is 0) and $\theta = \pi/4$ (the limit is $-1/(2\sqrt{2})$).

9.3.8

- Define $H : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$H(s,t) = (x,y), \qquad x = st, \quad y = s+t.$$

  Its differential matrix is

$$[dH(s,t)] = \begin{pmatrix} \partial x/\partial s & \partial x/\partial t \\ \partial y/\partial s & \partial y/\partial t \end{pmatrix} = \begin{pmatrix} t & s \\ 1 & 1 \end{pmatrix}.$$

- We have $G = F \circ H$. Therefore, by the chain rule,

$$dG(s,t) = dF(x,y)dH(s,t).$$

- Writing $F(x,y) = (f_1(x,y), f_2(x,y))$ and $G(s,t) = (g_1(s,t), g_2(s,t))$, we have the following differential matrices:

$$[dF] = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix}, \qquad [dG] = \begin{pmatrix} \partial g_1/\partial s & \partial g_1/\partial t \\ \partial g_2/\partial s & \partial g_2/\partial t \end{pmatrix}.$$

- It follows that

$$\begin{aligned} \begin{pmatrix} \partial g_1/\partial s & \partial g_1/\partial t \\ \partial g_2/\partial s & \partial g_2/\partial t \end{pmatrix} &= \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix} \begin{pmatrix} t & s \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} t\partial f_1/\partial x + \partial f_1/\partial y & s\partial f_1/\partial x + \partial f_1/\partial y \\ t\partial f_2/\partial x + \partial f_2/\partial y & s\partial f_2/\partial x + \partial f_2/\partial y \end{pmatrix}. \end{aligned}$$

- The direct chain-rule calculation for g_1 goes like this:

$$\begin{aligned} \frac{\partial g_1}{\partial s} &= \frac{\partial f_1}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f_1}{\partial y}\frac{\partial y}{\partial s} = t\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y}, \\ \frac{\partial g_1}{\partial t} &= \frac{\partial f_1}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_1}{\partial y}\frac{\partial y}{\partial t} = s\frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial y}, \end{aligned}$$

  with the same computation for g_2.

9.4.2

- The gradient of f is

$$df(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x + y, 3y^2 + x).$$

- At $(1,1)$, we get

$$df(1,1) = (3,4).$$

- The direction in which f is increasing at the greatest rate is

$$\frac{df(1,1)}{\|df(1,1)\|} = \left(\frac{3}{5}, \frac{4}{5}\right).$$

  Similarly, f is decreasing at the greatest rate in the direction

$$-\frac{df(1,1)}{\|df(1,1)\|} = \left(-\frac{3}{5}, -\frac{4}{5}\right).$$

- The directional derivative of f is 0 in directions orthogonal to the gradient. These directions are

$$\left(\frac{4}{5}, -\frac{3}{5}\right) \quad \text{and} \quad \left(-\frac{4}{5}, \frac{3}{5}\right).$$

9.4.3

- The tangent vector to the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ at $\gamma(t)$ is in the direction

$$d\gamma(t) = \left(3t^2, -1/t^2, 2e^{2t-2}\right).$$

- At $t = 1$, we have

$$\gamma(1) = (1, 1, 1), \qquad d\gamma(1) = (3, -1, 2).$$

- The tangent line to the curve is the line though $\gamma(1)$ in the direction $d\gamma(1)$, which has the equation

$$(x, y, z) = (1, 1, 1) + t(3, -1, 2)$$

  or

$$x = 1 + 3t, \qquad y = 1 - t, \qquad z = 1 + 2t.$$

- Alternatively, the unit tangent vector to the curve at $(1, 1, 1)$ is

$$T = \frac{1}{\sqrt{14}}(3, -1, 2),$$

  and a parametrization of the tangent line by arc-length s is

$$(x, y, z) = (1, 1, 1) + \frac{s}{\sqrt{14}}(3, -1, 2).$$