Real Analysis

Math 125B, Spring 2013

Midterm I: Solutionss

1. Prove that each of the following functions $f, g, h : [0, 1] \to \mathbb{R}$ is Riemann integrable on $[0, 1]$.

(a)

$$f(x) = x^2 \ln(2 + \sin x) + \cos(1 + e^x).$$

(b)

$$g(x) = \begin{cases} 1/(n+1) & \text{if } 1/2^{n+1} < x \leq 1/2^n \text{ (for } n = 0, 1, 2 \ldots), \\ 0 & \text{if } x = 0. \end{cases}$$

(c)

$$h(x) = \begin{cases} \operatorname{sgn}(\sin(1/x)) & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Here, sgn is the sign-function:

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Solution.

- (a) The function is continuous on $[0, 1]$ since it is a sum, product and composition of continuous functions ($\ln(2 + \sin x)$ is well-defined and continuous because $2 + \sin x \geq 1$), so it is Riemann integrable on $[0, 1]$.
- (b) The function is monotone increasing on $[0, 1]$, so it is Riemann integrable.
- (c) The function has discontinuities at $x = 0$ and $x = 1/(n\pi)$ for $n \in \mathbb{N}$. It is Riemann integrable on $[a, 1]$ for every $0 < a < 1$, since it has finitely many discontinuities there, and bounded on $[0, 1]$. It follows from a result in the homework that the function is Riemann integrable on $[0, 1]$.

2. Suppose that $f : [0,1] \to \mathbb{R}$ is a bounded function with upper and lower integrals

$$U(f) = \overline{\int_0^1} f = 1, \qquad L(f) = \underline{\int_0^1} f = 0.$$

(a) Prove that for every $\epsilon > 0$ there exists a partition P of $[0,1]$ such that the difference between the upper and lower sums of f on P satisfies

$$1 \le U(f;P) - L(f;P) < 1 + \epsilon.$$

(b) Does there have to be a partition P such that $U(f;P) - L(f;P) = 1$?

Solution.

- (a) Since $U(f) = \inf U(f;P)$ and $L(f) = \sup L(f;P)$, we have for every partition P that

  $$U(f;P) \ge U(f) = 1, \qquad L(f;P) \le L(f) = 0,$$

  so $U(f;P) - L(f;P) \ge 1$.

- Let $\epsilon > 0$. By the definition of the sup and inf, there are partitions Q, R of $[0,1]$ such that

  $$U(f;Q) < 1 + \frac{\epsilon}{2}, \qquad L(f;R) > -\frac{\epsilon}{2}.$$

  If P is a common refinement of Q and R, then $U(f;P) \le U(f;Q)$ and $L(f;P) \ge L(f;R)$, which implies that

  $$U(f;P) - L(f;P) \le U(f;Q) - L(f;R) < 1 + \epsilon.$$

- (b) No there doesn't. For example, consider the following slight modification of the Dirichlet function:

  $$f(x) = \begin{cases} 2 & \text{if } x = 0 \\ 1 & \text{if } x \in (0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in (0,1] \setminus \mathbb{Q} \end{cases}$$

  Then $U(f) = 1$ and $L(f) = 0$ since it differs from the Dirichlet function at a single point. For every partition $P = \{I_1, \dots, I_n\}$ of $[0,1]$, we have $L(f;P) = 0$ and $U(f;P) = 1 + |I_1| > 1$. So there is no partition P for which $U(f;Q) - L(f;P)$ attains its infimum of 1.

3. (a) State the theorem for integration by parts.

For parts (b) and (c), suppose that $f : [a, b] \to \mathbb{R}$ is a function which satisfies the following conditions:

1. f is continuous on $[a, b]$ and $f(a) = f(b) = 0$;
2. f is differentiable on $[a, b]$ with continuous derivative $f' : [a, b] \to \mathbb{R}$;
3. f' is differentiable in (a, b) with integrable derivative f''.

(b) Prove that

$$\int_a^b f f'' \, dx \leq 0.$$

(c) For what functions f that satisfy the conditions stated above is the integral in (b) equal to 0?

Solution.

- (a) Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable in (a, b) with integrable derivatives $f', g' : (a, b) \to \mathbb{R}$. Then

$$\int_a^b f g' \, dx = f(b)g(b) - f(a)g(a) - \int_a^b f' g \, dx.$$

- (b) The stated conditions allow us to apply integration by parts to f, $g = f'$ to get

$$\int_a^b f f'' \, dx = f(b)f'(b) - f(a)f'(a) - \int_a^b (f')^2 \, dx.$$

  Using the assumption that $f(a) = f(b) = 0$, the fact that $(f')^2 \geq 0$, and the positivity of the integral, we get

$$\int_a^b f f'' \, dx = -\int_a^b (f')^2 \, dx \leq 0.$$

- (c) If the intgeral is zero, then

$$\int_a^b (f')^2 \, dx = 0.$$

  Since $(f')^2$ is positive and continuous, a result from a homework problem implies that $f' = 0$, so $f = \text{constant}$. Since $f(a) = 0$ and f is continuous on $[a, b]$, it follows that $f = 0$ is the only such function.

4. Prove that

$$0 \leq \int_0^{\pi/2} \sin\left(\sin x\right)\, dx \leq 1.$$

Solution.

- We have $0 \leq \sin x \leq 1$ for $0 \leq x \leq \pi/2$, so $\sin(\sin x) \geq 0$. By the positivity of the integral

$$\int_0^{\pi/2} \sin(\sin x)\, dx \geq 0.$$

- Using the inequality $\sin\theta \leq \theta$ for $\theta \geq 0$, the monotonicity of the integral, and the fundamental theorem of calculus, we get that

$$\int_0^{\pi/2} \sin\left(\sin x\right)\, dx \leq \int_0^{\pi/2} \sin x\, dx = \left[-\cos x\right]_0^{\pi/2} = 1.$$

- The numerical value of the integral, computed by MATLAB, is

$$\int_0^{\pi/2} \sin\left(\sin x\right)\, dx = 0.8932437409750...$$

  The inverse symbolic calculator located at

  http://isc.carma.newcastle.edu.au/

  doesn't find a closed form representation for a number with this decimal expansion, so there's presumably no simple explicit expression for the value of the integral (though from tables, it looks like it might be expressible in terms of Lommel functions).