Midterm 2: Sample question solutions
Math 125B: Winter 2013

1. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$f(x, y, z) = \left(x^2 + yz, \sin(xyz) + z\right).$$

(a) Why is f differentiable on $\mathbb{R}^3$? Compute the Jacobian matrix of f at $(x, y, z) = (-1, 0, 1)$.
(b) Are there any directions in which the directional derivative of f at $(-1, 0, 1)$ is zero? If so, find them.

Solution.

- (a) The partial derivatives of the component functions of f exist and are continuous on $\mathbb{R}^3$, so f is differentiable on $\mathbb{R}^3$.

- Explicitly, we write $f = (f_1, f_2)$ where $f_1, f_2 : \mathbb{R}^3 \to \mathbb{R}$ are given by

  $$f_1(x, y, z) = x^2 + yz, \qquad f_2(x, y, z) = \sin(xyz) + z.$$

  The partial derivatives are

  $$\frac{\partial f_1}{\partial x}(x, y, z) = 2x, \qquad \frac{\partial f_1}{\partial y}(x, y, z) = z, \qquad \frac{\partial f_1}{\partial z}(x, y, z) = y,$$
  $$\frac{\partial f_2}{\partial x}(x, y, z) = yz\cos(xyz), \qquad \frac{\partial f_2}{\partial y}(x, y, z) = xz\cos(xyz),$$
  $$\frac{\partial f_3}{\partial z}(x, y, z) = xy\cos(xyz) + 1,$$

  and the Jacobian matrix of f is

  $$[df] = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \end{pmatrix}$$

- In particular,

  $$\frac{\partial f_1}{\partial x}(-1, 0, 1) = -2, \qquad \frac{\partial f_1}{\partial y}(-1, 0, 1) = 1, \qquad \frac{\partial f_1}{\partial z}(-1, 0, 1) = 0,$$
  $$\frac{\partial f_2}{\partial x}(-1, 0, 1) = 0, \qquad \frac{\partial f_2}{\partial y}(-1, 0, 1) = -1, \qquad \frac{\partial f_3}{\partial z}(-1, 0, 1) = 1.$$

Therefore, the Jacobian matrix of f at $(-1,0,1)$ is

$$[df(-1,0,0)] = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

- (b) The directional derivative of f at $(-1,0,1)$ in the direction h is

  $$D_h f(-1,0,1) = df(-1,0,1)h.$$

  If $h = (a,b,c)$, then the directional derivative has components

  $$[D_h f(-1,0,1)] = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2a+b \\ -b+c \end{pmatrix}$$

- This is zero if $b = 2a$ and $c = b$, or $h = k(1,2,2)$ where k is an arbitrary constant. Thus, normalizing h to unit vector, with a convenient choice of sign, the directional derivative of f at $(-1,0,1)$ is zero in the direction

  $$h = \frac{1}{3}(1,2,2).$$

2. Suppose that $f : \mathbb{R} \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are defined by

$$f(t) = (t, t^2, t^3), \qquad g(x, y, z) = x^2 e^{yz},$$

and $h = g \circ f : \mathbb{R} \to \mathbb{R}$ is their composition.
(a) Use the chain rule to compute $h'(1)$.
(b) Find $h(t)$ and compute $h'(1)$ directly.

Solution.

- (a) Note that f is differentiable on $\mathbb{R}$, since each of its component functions is differentiable, and g is differentiable on $\mathbb{R}^3$ since its partial derivatives exist and are continuous.

- The Jacobian matrix of f at $t = 1$ is

$$[df(1)] = \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}\Bigg|_{t=1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

- We have $f(1) = (1, 1, 1)$, and the Jacobian matrix of g at $(1, 1, 1)$ is

$$[dg(f(1))] = \begin{pmatrix} 2xe^{yz} & x^2 ze^{yz} & x^2 ye^{yz} \end{pmatrix}\Big|_{(x,y,z)=(1,1,1)} = \begin{pmatrix} 2e & e & e \end{pmatrix}.$$

- By the chain rule, $h = g \circ f$ is differentiable on $\mathbb{R}$ and

$$h'(1) = [dg(f(1))][df(1)] = \begin{pmatrix} 2e & e & e \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2e + 2e + 3e = 7e.$$

- We have

$$h(t) = g(t, t^2, t^3) = t^2 e^{t^5}.$$

  Thus, by the product and chain rules from one-variable calculus,

$$h'(1) = \left(2t \cdot e^{t^5} + t^2 \cdot 5t^4 e^{t^5}\right)\Big|_{t=1} = 7e.$$

3. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} x^{4/3}\sin(y/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Where is f is differentiable?

Solution.

- The function f is differentiable at every point of $\mathbb{R}^2$.
- By the chain and product rules, the partial derivatives of f,

$$\frac{\partial f}{\partial x}(x,y) = \frac{4}{3}x^{1/3}\sin(y/x) - yx^{-2/3}\cos(y/x),$$
$$\frac{\partial f}{\partial y}(x,y) = x^{1/3}\cos(y/x),$$

  exist and are continuous in the open set

$$U = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}.$$

  Therefore f is differentiable in U.

- We claim that the partial derivatives of f also exist if $x = 0$ and are equal to 0.
- For the partial derivative with respect to x, we have

$$\begin{aligned} \frac{\partial f}{\partial x}(0,y) &= \lim_{h\to 0}\frac{f(h,y) - f(0,y)}{h} \\ &= \lim_{h\to 0}\frac{h^{4/3}\sin(y/h)}{h} \\ &= \lim_{h\to 0} h^{1/3}\sin(y/h) \\ &= 0, \end{aligned}$$

  where we use the 'squeeze' theorem and the inequality

$$|h^{1/3}\sin(y/h)| \leq |h|^{1/3}.$$

- Since $f(0, y) = 0$ for every $y \in \mathbb{R}$, for the partial derivative with respect to y, we have
$$\frac{\partial f}{\partial y}(0, y) = \frac{d}{dy} f(0, y) = 0.$$
.

- It follows that *if* f is differentiable at $(0, y)$, then its derivative
$$[df(0, y)] = (\partial f/\partial x(0, y), \partial f/\partial y(0, y))$$
must be zero. We prove that this is indeed the case from the definition of the derivative.

- Consider the difference between f and its affine approximation at $(0, y)$:
$$r(h, k) = f(h, y + k) - f(0, y) - df(0, y) \cdot (h, k).$$
To prove that f is differentiable at $(0, y)$ with derivative $df(0, y)$, we need to show that
$$\lim_{(h,k)\to(0,0)} \frac{|r(h, k)|}{\|(h, k)\|} = 0.$$

- Supposing that $[df(0, y)] = (0, 0)$ and using the fact that $f(0, y) = 0$, we get $r(h, k) = f(h, y + k)$. Therefore,
$$|r(h, k)| \leq |h|^{4/3},$$
since either $h = 0$ and $r(h, k) = 0$, or $h \neq 0$ and
$$|r(h, k)| = \left|h^{4/3} \sin((y + k)/h)\right| \leq |h|^{4/3}.$$
It follows that
$$\frac{|r(h, k)|}{\|(h, k)\|} \leq \frac{|h|^{4/3}}{(h^2 + k^2)^{1/2}} \leq |h|^{1/3} \to 0 \qquad \text{as } (h, k) \to (0, 0).$$

- This proves that f is differentiable at $(0, y)$ with derivative $df(0, y) = 0$.

Remark. Note that $\partial f/\partial y$ is continuous, but $\partial f/\partial x$ is not continuous at $(0, y)$, except when $y = 0$. This function is differentiable even though it has a discontinuous partial derivative, so while the continuity of partial derivatives is a sufficient condition for differentiability, it isn't a necessary one.

4. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$. If $A : \mathbb{R}^m \to \mathbb{R}^p$ is a linear map, prove from the definition of the derivative that $Af : \mathbb{R}^n \to \mathbb{R}^p$ is differentiable at x and find its derivative. (You can assume that $\|Ah\| \le M\|h\|$ for some constant M. See p. 212 of the text)

Solution.

- From the definition of the derivative,

$$f(x+h) = f(x) + df(x)h + r(h)$$

  where the derivative $df(x) : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map and $r(h) = o(h)$ as $h \to 0$, meaning that

$$\lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} = 0.$$

- It follows from the linearity of A that

$$Af(x+h) = Af(x) + A\,df(x)h + Ar(h),$$

  and $A\,df(x) : \mathbb{R}^n \to \mathbb{R}^p$ is a linear map (i.e., $df(x)$ followed by A). Moreover, $Ar(h) = o(h)$ as $h \to 0$ since

$$\frac{\|Ar(h)\|}{\|h\|} \le M\frac{\|r(h)\|}{\|h\|} \to 0 \qquad \text{as } h \to 0.$$

- This proves that Af is differentiable at x with derivative

$$d(Af)(x) = A\,df(x).$$

Remark. This problem is a special case of the chain rule where one of the functions is linear, so the function and its derivative are equal.

5. The mean value theorem from one-variable calculus states that if a function $f : [a, b] \to \mathbb{R}$ is continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b), then there is a point $a < c < b$ such that

$$f(b) - f(a) = (b - a)f'(c).$$

Does this theorem remain true for a vector-valued function $f : [a, b] \to \mathbb{R}^2$?

Solution.

- It doesn't remain true. The reason is that if $f = (f_1, f_2)$, we may need to use different points c_1, c_2 to satisfy the mean value theorem for the real-valued component functions $f_1, f_2 : [a, b] \to \mathbb{R}$.

- To give an explicit counter-example, define $f : [0, 1] \to \mathbb{R}^2$ by

  $$f(x) = \left(x(1 - x), x^2(1 - x)\right), \quad f_1(x) = x(1 - x), \quad f_2(x) = x^2(1 - x).$$

  Then f is continuous on $[0, 1]$ and differentiable in $(0, 1)$, since each component function is, and $f(0) = f(1) = (0, 0)$. However, $f_1'(c_1) = 0$ if and only if $c_1 = 1/2$, while $f_2'(c_2) = 0$ at an interior point if and only if $c_2 = 2/3$. Thus, there is no point $0 < c < 1$ such that $f'(c) = (0, 0)$.

Optional remark. Frequently, we use the mean value theorem for a real-valued function in the following way: if $|f'(x)| \leq M$ for $a < x < b$, then $|f(b) - f(a)| \leq M|b - a|$. Although the mean value theorem itself fails for vector-valued functions, there is a useful generalization of this estimate that can often be used instead. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable, and $a, b \in \mathbb{R}^n$, then the chain rule implies that for $t \in \mathbb{R}$,

$$\frac{d}{dt} f\left(a + t(b - a)\right) = df\left(a + t(b - a)\right)(b - a).$$

Suppose that $\|df\left(a + t(b - a)\right)\| \leq M$ for $0 \leq t \leq 1$, where $\|A\| \geq 0$ denotes the norm of a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ (i.e., the smallest constant such that $\|Ah\| \leq \|A\|\|h\|$ for all $h \in \mathbb{R}^n$). Then the fundamental theorem of calculus implies that

$$f(b) - f(a) = \int_0^1 \frac{d}{dt} f\left(a + t(b - a)\right)\, dt = \int_0^1 df\left(a + t(b - a)\right)(b - a)\, dt,$$

and this gives the estimate

$$\|f(b) - f(a)\| \leq \int_0^1 \|df\left(a + t(b - a)\right)(b - a)\|\, dt \leq M\|b - a\|.$$