Real Analysis

Math 125B, Spring 2013

Midterm II: Solutions

1. Define curves $\beta : \mathbb{R} \to \mathbb{R}^3$, $\gamma : \mathbb{R} \to \mathbb{R}^2$ and a function $f : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\beta(t) = (t, t^2, t^3), \qquad \gamma(t) = (t^5, t^7), \qquad f(x, y, z) = (x^3 y, y^2 z).$$

(a) Compute the differential matrices $[d\beta(1)]$ and $[d\gamma(1)]$.
(b) Compute the differential matrix $[df(1, 1, 1)]$ of f at $(1, 1, 1)$.
(c) Verify that $\gamma = f \circ \beta$ and $[d\gamma(1)] = [df(\beta(1))][d\beta(1)]$. Can you give a geometrical interpretation of df in terms of tangent vectors?

Solution.

- (a) We write $[d\beta]$, $[d\gamma]$ in vector notation as a column vectors, to get

$$[d\beta(1)] = \left.\begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}\right|_{t=1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad [d\gamma(1)] = \left.\begin{pmatrix} 5t^4 \\ 7t^6 \end{pmatrix}\right|_{t=1} = \begin{pmatrix} 5 \\ 7 \end{pmatrix}.$$

- (b) The differential matrix is

$$\begin{aligned} [df(1, 1, 1)] &= \left.\begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial z \\ \partial f_2/\partial x & \partial f_2/\partial y & \partial f_2/\partial z \end{pmatrix}\right|_{(x,y,z)=(1,1,1)} \\ &= \left.\begin{pmatrix} 3x^2 y & x^3 & 0 \\ 0 & 2yz & y^2 \end{pmatrix}\right|_{(x,y,z)=(1,1,1)} \\ &= \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}. \end{aligned}$$

- (c) We have $(f \circ \beta)(t) = (t^3 \cdot t^2, t^4 \cdot t^3) = \gamma(t)$, $\beta(1) = (1, 1, 1)$, and

$$[df(\beta(1))][d\beta(1)] = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} = [d\gamma(1)].$$

- The geometrical interpretation of $df(x)$ is that if β is a curve through x, then f maps β to the curve $\gamma = f \circ \beta$ through $f(x)$, and $df(x)$ maps the tangent vector of β at x to the tangent vector of γ at $f(x)$.

2. (a) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $h \in \mathbb{R}^n$. Define the directional derivative $D_h f(x)$ of f in the direction h at a point $x \in \mathbb{R}^n$.
(b) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that the directional derivative $D_{(a,b)} f(0,0)$ of f at the point $(0,0)$ exists in every direction (a,b) and compute it.
(c) Is the function f in (b) differentiable at $(0,0)$? Use the result of (b) to justify your answer.

Solution.

- (a) The directional derivative is defined by

  $$D_h f(x) = \frac{d}{dt} f(x+th)\bigg|_{t=0}.$$

  (b) The directional derivative of f in the direction $(a,b) \neq (0,0)$ is

  $$\begin{aligned} D_{(a,b)} f(0,0) &= \frac{d}{dt} f(ta,tb)\bigg|_{t=0} \\ &= \frac{d}{dt}\left(\frac{ta^3}{a^2+b^2}\right)\bigg|_{t=0} \\ &= \frac{a^3}{a^2+b^2}. \end{aligned}$$

  Note that the expression for $f(ta,tb)$ is valid at $t = 0$, since $f(0,0) = 0$.

- (c) If f was differentiable at $(0,0)$, then the directional derivative

  $$D_{(a,b)} f(0,0) = df(0,0)(a,b)$$

  would be a linear function of the direction (a,b). From (b), the directional derivative at $(0,0)$ is not linear in (a,b), so f is not differentiable at $(0,0)$.

- In fact, if f was differentiable, its differential matrix would be

$$[df(0,0)] = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right) = (1,0), \qquad ,$$

and its directional derivative would be

$$D_{(a,b)}f(0,0) = (1,0)\begin{pmatrix} a \\ b \end{pmatrix} = a,$$

which it isn't.

Remark. As this example shows, the directional derivatives of a function may exist in every direction and be a nonlinear function of the direction. In that case, however, the function can't be differentiable.

3. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(r, \theta) = (x, y), \qquad x = r\cos\theta, \quad y = r\sin\theta.$$

(a) What is the differential matrix $[df(r, \theta)]$ of f?
(b) Suppose that $g(x, y)$ is a differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$ and

$$h = g \circ f : \mathbb{R}^2 \to \mathbb{R}, \qquad h(r, \theta) = g(r\cos\theta, r\sin\theta).$$

Use the result of (a) and the chain rule to express $\partial h/\partial r$ and $\partial h/\partial\theta$ in terms of $\partial g/\partial x$ and $\partial g/\partial y$.
(c) Compute $\partial h/\partial r$ and $\partial h/\partial\theta$ if $g(x, y) = x^2 - y^2$.

Solution.

- (a) The differential matrix of f is

$$[df(r, \theta)] = \begin{pmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

- (b) Since $h = g \circ f$, the chain rule implies that

$$[dh(r, \theta)] = [dg(x, y)][df(r, \theta)].$$

  This gives

$$\begin{aligned} \left(\frac{\partial h}{\partial r}, \frac{\partial h}{\partial\theta}\right) &= \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= \left(\cos\theta\frac{\partial g}{\partial x} + \sin\theta\frac{\partial g}{\partial y}, -r\sin\theta\frac{\partial g}{\partial x} + r\cos\theta\frac{\partial g}{\partial y}\right), \end{aligned}$$

  so

$$\frac{\partial h}{\partial r} = \cos\theta\frac{\partial g}{\partial x} + \sin\theta\frac{\partial g}{\partial y}, \qquad \frac{\partial h}{\partial\theta} = -r\sin\theta\frac{\partial g}{\partial x} + r\cos\theta\frac{\partial g}{\partial y}$$

- The direct calculation goes like this:

$$\begin{aligned} \frac{\partial h}{\partial r} &= \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial g}{\partial x} + \sin\theta\frac{\partial g}{\partial y}, \\ \frac{\partial h}{\partial\theta} &= \frac{\partial g}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial\theta} = -r\sin\theta\frac{\partial g}{\partial x} + r\cos\theta\frac{\partial g}{\partial y}. \end{aligned}$$

- (c) If $g(x, y) = x^2 - y^2$, then

$$\frac{\partial g}{\partial x} = 2x = 2r\cos\theta, \qquad \frac{\partial g}{\partial x} = -2y = -2r\sin\theta$$

and

$$\frac{\partial h}{\partial r} = 2r\cos^2\theta - 2r\sin^2\theta, \qquad \frac{\partial h}{\partial \theta} = -4r^2\cos\theta\sin\theta.$$

Remark. This problem gives the formula for transforming partial derivatives with respect to Cartesian coordinates into partial derivatives with respect to polar coordinates. In practice, we often use the same symbol for related functions that are expressed in terms of (x, y) or (r, θ) and write $h = g$. This simplifies the notation but can lead to confusion if you don't understand what's going on.

4. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } x, y \in \mathbb{Q} \text{ are both rational,} \\ 0 & \text{if at least one of } x,\ y \text{ is irrational.} \end{cases}$$

(a) Prove that f is differentiable at $(0,0)$. What is its derivative?

(b) Prove that f is not differentiable at any point $(x,y) \neq (0,0)$.

(c) At what points $(x,y) \in \mathbb{R}^2$ does $\partial f/\partial x$ exist, and at what points does $\partial f/\partial y$ exist? What are the values of the partial derivatives at points where they do exist?

Solution.

- (a) The derivative is $df(0,0) = 0$. To prove this, we consider the limit

  $$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - df(0,0)(h,k)|}{\|(h,k)\|} = \lim_{(h,k)\to(0,0)} \frac{|f(h,k)|}{\|(h,k)\|}.$$

  If h, k are rational, then

  $$|f(h,k)| = h^2 + k^2 = \|(h,k)\|^2,$$

  and if h or k is irrational, then $f(h,k) = 0$. In either case, we have

  $$\frac{|f(h,k)|}{\|(h,k)\|} \leq \|(h,k)\| \to 0 \qquad \text{as } (h,k) \to 0,$$

  so the limit defining the derivative is zero (by the 'squeeze' theorem), and f is differentiable at $(0,0)$ with zero derivative.

- (b) The function f is discontinuous at every point $(x,y) \neq (0,0)$, so it is not differentiable away from the origin.

- If x, y are both rational, then $f(x,y) = x^2 + y^2 > 0$, but every neighborhood of (x,y) contains irrational points (x', y') with $f(x', y') = 0$. It follows that the ϵ-δ definition of continuity fails if $0 < \epsilon < x^2 + y^2$.

- If at least one of x, y is irrational, then $f(x,y) = 0$, but every neighborhood of (x,y) contains rational points with $f(x', y') = x'^2 + y'^2$. Moreover, there are rational points in every neighborhood with $x'^2 + y'^2 > x^2 + y^2$. It then also follows that the ϵ-δ definition of continuity fails if $0 < \epsilon < x^2 + y^2$.

- (c) The differentiability of f at $(0,0)$ implies that both its partial derivatives exist and are equal to 0.
- If $x \in \mathbb{Q}$ is non-zero and rational, then

  $$f(x,y) = \begin{cases} x^2 + y^2 & \text{if } y \in \mathbb{Q}, \\ 0 & \text{if } y \notin \mathbb{Q}. \end{cases}$$

  This function is discontinuous at every $y \in \mathbb{R}$, so $\partial f/\partial y$ doesn't exist.
- If $x \notin \mathbb{Q}$, then $f(x,y) = 0$ for all $y \in \mathbb{R}$, so $\partial f/\partial y = 0$.
- The argument for the partial derivatives with respect to x is the same.
- We conclude that the partial derivative with respect to x exists if $(x,y) = (0,0)$ or $y \notin \mathbb{Q}$, and the partial derivative with respect to y exists if $(x,y) = (0,0)$ or $x \notin \mathbb{Q}$. The partial derivatives are zero whenever they exist.

Remark. The partial derivatives of f are undefined at a dense set of points in every neighborhood of $(0,0)$, so they are certainly not continuous at $(0,0)$. Nevertheless f is differentiable at $(0,0)$.