Problem Set 3
Math 125B
Winter 2013

1. Problem 5.2.10 in the text.

2. Let $K \subset [0,1]$ be the standard Cantor set described in class. Let $f : [0,1] \to \mathbb{R}$ be the characteristic function of K, defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in K \\ 0 & \text{if } x \notin K \end{cases}$$

(a) Prove that K is the set of discontinuities of f.
(b) Prove that f is Riemann integrable. (Note: you're not allowed to assume the Lebesgue criterion for Riemann integrability, but you can use the theorems about the existence of Riemann integrals that we proved in class.)

3. Suppose that $F : [a,b] \to \mathbb{R}$ has a continuous derivative $F' : [a,b] \to \mathbb{R}$. Use the second fundamental theorem of calculus (Theorem 5.3.3 in the text) to deduce the first fundamental theorem, that

$$\int_a^b F' = F(b) - F(a).$$

How is your conclusion weaker than the one stated in Theorem 5.3.1 of the text?

Sample Midterm Questions

Justify your answers with appropriate theorems.

1. Let

$$L(x) = \int_1^x \frac{1}{t}\, dt.$$

For what values of x is $L(x)$ defined as an oriented Riemann integral?

2. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Compute the upper and lower integrals of f. Is f Riemann integrable on $[0,1]$?
Hint. You can assume the summation formula

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1).$$

3. (a) State the second part of the fundamental theorem of calculus (i.e., the direction that says "the derivative of the integral is the original function").
(b) Let $h : [-1,1] \to \mathbb{R}$ be the Heaviside step function

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Evaluate $H : [-1,1] \to \mathbb{R}$ where

$$H(x) = \int_{-1}^{x} h(x)\, dx.$$

Is H continuous on $[-1,1]$? Is H differentiable on $[-1,1]$? Does H satisfy the conclusions of the fundamental theorem of calculus you stated in (a)?

4. Define $F, f : [0,1] \to \mathbb{R}$ by

$$F(x) = x^2 \sin\left(\frac{1}{x}\right), \qquad f(x) = -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$$

if $0 < x \leq 1$ and $F(0) = f(0) = 0$. Verify that that $F'(x) = f(x)$ for $0 < x < 1$ and evaluate

$$\int_0^1 f(x)\, dx.$$

5. Suppose that $f : [0,1] \to \mathbb{R}$ is a continuous positive function ($f \geq 0$) such that

$$\int_0^1 f = 0.$$

Prove that $f = 0$. Does this result remain true if f is only assumed to be integrable?