This sequence certainly has limit 0 and so, by Theorem 5.1.8, the Riemann integral exists. To find what it is, we need a formula for the sum $\sum_{k=1}^{n} k^2$. Such a formula exists. In fact, it can be proved by induction (Exercise 5.1.3) that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Thus,

$$U(f, P_n) = \frac{n(n+1)(2n+1)}{6n^3} = \frac{(1+1/n)(2+1/n)}{6}.$$

This expression has limit 1/3 as $n \to \infty$ and so $\int_0^1 x^3\,dx = 1/3$.

Exercise Set 5.1

1. Find the upper sum $U(f, P)$ and lower sum $L(f, P)$ if $f(x) = 1/x$ on $[1, 2]$ and P is the partition of $[1, 2]$ into four subintervals of equal length.
2. Prove that $\int_0^1 x\,dx$ exists by computing $U(f, P_n)$ and $L(f, P_n)$ for the function $f(x) = x$ and a partition P_n of $[0, 1]$ into n equal subintervals. Then show that condition (5.1.7) of Theorem 5.1.8 is satisfied. Calculate the integral by taking the limit of the upper sums. Hint: Use Exercise 1.2.9.
3. Prove by induction that
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$
4. Prove that $\int_0^a x^2\,dx = \frac{a^3}{3}$ by expressing this integral as a limit of Riemann sums and finding the limit.
5. Let f be the function on $[0, 1]$ which is 0 at every rational number and 1 at every irrational number. Is this function integrable on $[0, 1]$? Prove that your answer is correct by using the definition of the integral.
6. Prove that the upper sum $U(f, P)$ for a partition of $[a, b]$ and a bounded function f on $[a, b]$ is the least upper bound of the set of all Riemann sums for f and P.
7. Finish the proof of Theorem 5.1.4 by showing that if the theorem is true when only one element is added to P to obtain Q, then it is also true no matter how many elements need to be added to P to obtain Q.
8. Suppose m and M are lower and upper bounds for f on $[a, b]$; in other words, $m \le f(x) \le M$ for all $x \in [a, b]$. Prove that
$$m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a).$$
What conclusion about $\int_a^b f(x)\,dx$ do you draw from this if the integral exists?

9. If k is a constant and $[a, b]$ is a bounded interval, prove that k is integrable on $[a, b]$ and
$$\int_a^b k\,dx = k(b-a).$$
10. If f is a bounded function on $[a, b]$ and $P = \{x_0 < x_1 < \cdots < x_n\}$ is a partition of $[a, b]$, show that
$$U(f, P) - L(f, P) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}),$$
where M_k is the sup and m_k the inf of f on $[x_{k-1}, x_k]$. What does this simplify to if P is a partition of $[a, b]$ into n equal subintervals?
11. Suppose f is any non-decreasing function on a bounded interval $[a, b]$. If P_n is the partition of $[a, b]$ into n equal subintervals, show that
$$U(f, P_n) - L(f, P_n) = (f(b) - f(a))\frac{b-a}{n}.$$
What do you conclude about the integrability of f?

5.2. Existence and Properties of the Integral

We first show that the integral exists for a large class of functions, a class which includes all the functions of interest to us in this course. We then show that the integral has the properties claimed for it in calculus courses.

Existence Theorems.

Theorem 5.2.1. *If f is a monotone function on a closed bounded interval $[a, b]$, then f is integrable on $[a, b]$.*

Proof. This was essentially stated as an exercise (Exercise 5.1.11) in the previous section. In that exercise, it is claimed that, if f is a non-decreasing function on $[a, b]$ and P_n is the partition of $[a, b]$ into n equal subintervals, then

$$U(f, P_n) - L(f, P_n) = (f(b) - f(a))\frac{b-a}{n}. \tag{5.2.1}$$

This implies that
$$\lim(U(f, P_n) - L(f, P_n)) = 0$$
and, by Theorem 5.1.8, this implies that the Riemann integral of f on $[a, b]$ exists.

In the case where f is non-increasing, the same proof works. The only difference is that $f(b) - f(a)$ is replaced by $f(a) - f(b)$ in (5.2.1). □

Theorem 5.2.2. *If f is a continuous function on a closed, bounded interval $[a, b]$, then f is integrable on $[a, b]$.*

But $\underline{\int}_a^c f(x)\,dx$ is the sup of all numbers of the form $L(Q,f)$ for Q a partition of $[a,c]$, and $L(f,P) = L(f,P') + L(L,P'')$. It follows from Theorem 1.5.7(c) that

$$\underline{\int}_a^b f(x)\,dx + \underline{\int}_b^c f(x)\,dx = \sup_{P'} L(P',f) + \sup_{P''} L(P'',f)$$
$$= \sup\{L(P',f) + L(P'',f)\} = \underline{\int}_a^c f(x)\,dx,$$

where P' and P'' range over arbitrary partitions of $[a,b]$ and $[b,c]$. This proves the theorem for lower integrals. The proof for upper integrals is essentially the same. □

This theorem has as a corollary the interval additivity property for the integral. The details of how this corollary follows from the above theorem are left to the exercises.

Corollary 5.2.9. *With f and $a \le b \le c$ as in the previous theorem, f is integrable on $[a,c]$ if and only if it is integrable on $[a,b]$ and on $[b,c]$. In this case,*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

A Stronger Existence Theorem. Another consequence of the interval additivity theorem (Theorem 5.2.8) is the following stronger version of the existence theorem for integrals of continuous functions (Theorem 5.2.2). The proof is left to the exercises.

Theorem 5.2.10. *If f is a bounded function on a closed bounded interval $[a,b]$ and f is continuous except at finitely many points of $[a,b]$, then f is integrable on $[a,b]$.*

Exercise Set 5.2

1. Show that if a function f on a bounded interval can be written in the form $g-h$ for functions g and h which are non-decreasing on $[a,b]$, then f is integrable on $[a,b]$.
2. If f is a bounded function defined on a closed bounded interval $[a,b]$ and if f is integrable on each interval $[a,r]$ with $a < r < b$, then prove that f is integrable on $[a,b]$ and
$$\int_a^b f(x)\,dx = \lim_{r \to b} \int_a^r f(x)\,dx.$$
Observe that the analogous result holds if $[a,r]$ is replaced by $[r,b]$ in the hypothesis and in the integral on the right and the limit is taken as $r \to a$. Hint: Use Theorem 5.2.8 and Exercise 5.1.8.
3. Prove Theorem 5.2.10. That is, prove that if f is a bounded function on a bounded interval $[a,b]$ and f is continuous except at finitely many points in $[a,b]$, then f is integrable on $[a,b]$. Hint: Use the preceding exercise, interval additivity, and an induction argument on the number of discontinuities.

4. Prove Corollary 5.2.5.
5. Prove Corollary 5.2.9.
6. Prove that $1 \le \int_{-1}^{1} \frac{1}{1+x^{2n}}\,dx \le 2$ for all $n \in \mathbb{N}$.
7. Prove that $\int_{-1}^{1} \frac{x^2}{1+x^{2n}}\,dx \le 2/3$ for all $n \in \mathbb{N}$.
8. If f is a bounded function defined on an interval I, then prove that

$$\sup_I |f| - \inf_I |f| \le \sup_I f - \inf_I f$$

   by using the triangle inequality, $|f(x)| - |f(y)| \le |f(x) - f(y)|$, and Theorem 1.5.10(d).
9. Prove that if f is integrable on $[a,b]$, then so is f^2. Hint: If $|f(x)| \le M$ for all $x \in [a,b]$, then show that

$$|f^2(x) - f^2(y)| \le 2M|f(x) - f(y)|$$

   for all $x, y \in [a,b]$. Use this to estimate $U(f^2,P) - L(f^2,P)$, for a given partition P, in terms of $U(f,P) - L(f,P)$.
10. Prove that if f and g are integrable on $[a,b]$, then so is fg. Hint: Write fg as the difference of two squares of functions you know are integrable and then use the previous exercise.
11. Give an example of a function f such that $|f|$ is integrable on $[0,1]$ but f is not integrable on $[0,1]$.
12. Prove Theorem 5.2.7.
13. Let $\{f_n\}$ be a sequence of integrable functions defined on a closed bounded interval $[a,b]$. If $\{f_n\}$ converges uniformly on $[a,b]$ to a function f, prove that f is integrable and

$$\int_a^b f(x)\,dx = \lim \int_a^b f_n(x)\,dx.$$

14. Is the function which is $\sin 1/x$ for $x \neq 0$ and 0 for $x = 0$ integrable on $[0,1]$? Justify your answer.

5.3. The Fundamental Theorems of Calculus

There are two fundamental theorems of calculus. Both relate differentiation to integration. In most calculus courses, the Second Fundamental Theorem is usually proved first and then used to prove the First Fundamental Theorem. Unfortunately, this results in a First Fundamental Theorem that is weaker than it could be. To prove the best possible theorems, one should give independent proofs of the two theorems. This is what we shall do.