The first of the above limits is $\lim_{a\to\infty} -1/2\ln(1+a^2) = -\infty$ while the second is $\lim_{b\to\infty} 1/2\ln(1+b^2) = \infty$. Neither of these converges and so the improper integral does not converge. However, the Cauchy principal value is

$$\lim_{a\to\infty}\int_{-a}^{a}\frac{x}{1+x^2}\,dx = \lim_{a\to\infty} 1/2(\ln a - \ln a) = 0.$$

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Exercise Set 5.4

1. Supply the details for the proof of Theorem 5.4.3.
2. Prove that $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ for all $a, b \in (0, +\infty)$.
3. Finish the proof of Theorem 5.4.4 by showing that $\lim_{x\to 0}\ln x = -\infty$. Hint: This follows easily from $\lim_{x\to\infty}\ln x = +\infty$ and properties of ln.
4. Prove part (b) of Theorem 5.4.7.
5. Using Definition 5.4.8 and the properties of exp prove the laws of exponents:

$$a^{x+y} = a^x a^y \quad \text{and} \quad a^{xy} = (a^x)^y.$$

6. Compute the derivative of a^x for each $a > 0$.
7. Find an antiderivative for a^x for each $a > 0$.
8. Prove Theorem 5.4.9.
9. For which values of $p > 0$ does the improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge? Justify your answer.
10. For which values of $p > 0$ does the improper integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converge? Justify your answer.
11. Show that $\displaystyle\int_{-\infty}^{\infty}\frac{\sin x}{1+x^2}\,dx$ converges. Can you tell what it converges to?
12. Does the improper integral $\displaystyle\int_0^1 \ln x\,dx$ converge? If so, what does it converge to?
13. Suppose that f and g are non-negative functions on $\mathbb{R}$ which are integrable on each finite interval $[a, b]$ and that $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Show that if the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ diverges, then so does the improper integral $\int_{-\infty}^{\infty} g(x)\,dx$.
14. Prove that if f is integrable on every interval $[a, b]$ on $\mathbb{R}$ and if f is an odd function, then f has Cauchy principal value 0.

15. Prove that the improper integral $\int_{-\infty}^{\infty} \frac{x^{1/3}}{\sqrt{1+x^2}}\, dx$ does not converge but that it has Cauchy principal value 0.
16. Prove that if f is an integrable function on every interval $[a, s)$ with $s < b$ and if $f(x) \geq 0$ on $[a, b]$, then the function $F(s) = \int_a^s f(x)\, dx$ is a non-decreasing function on $[a, b)$.