The Normed Vector Space $C(I)$. In mathematics we deal with a great many normed vector spaces. One that does not look at all like $\mathbb{R}^d$ is the space $C(I)$, where I is a closed bounded interval on the real line and $C(I)$ is the vector space of all continuous real-valued functions on I. Addition is pointwise addition of functions and scalar multiplication is multiplication of a function by a constant. It is easy to see that $C(I)$ is a vector space under these two operations (Exercise 7.1.10). There are many norms that can be put on this vector space, but perhaps the most useful is the sup norm, $||\cdot||_\infty$, defined by

$$||f||_\infty = \sup_I |f(x)|, \tag{7.1.3}$$

for $f \in C(I)$. The problem of showing that this is a norm is left to the exercises.

Exercise set 7.1

1. For the vectors $x = (1, 0, 2)$ and $y = (-1, 3, 1)$ in $\mathbb{R}^3$ find
   (a) $2x + y$;
   (b) $x \cdot y$;
   (c) $||x||$ and $||y||$;
   (d) the cosine of the angle between x and y;
   (e) the distance from x to y.
2. Using only the properties listed in Theorem 7.1.1, prove that if u, v, w are vectors in a vector space and $u + w = v + w$, then $u = v$.
3. Using only the properties listed in Theorem 7.1.1, prove that if u is a vector in a vector space, a is a scalar, and $au = 0$, then either $a = 0$ or $u = 0$.
4. Prove Theorem 7.1.4.
5. Prove the second form of the triangle inequality. That is, prove that
$$|||x|| - ||y||| \le ||x - y||$$
   holds for any pair of vectors x, y in a normed vector space. Hint: Use the first form (Theorem 7.1.10(a)) to prove the second form.
6. Prove that equality holds in the Cauchy-Schwarz inequality (Theorem 7.1.8) if and only if one of the vectors u, v is a scalar multiple of the other.
7. For a norm on a vector space X, defined by an inner product as in Definition 7.1.7, prove that the parallelogram law,
$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2,$$
   holds for all $x, y \in X$.
8. Prove that $||\cdot||_\infty$, as defined in Definition 7.1.11, is a norm on $\mathbb{R}^d$.
9. Prove Theorem 7.1.13
10. Prove that the space $C(I)$, defined in the previous subsection, is a vector space.
11. Prove that the sup norm as defined in (7.1.3) is really a norm on $C(I)$.

12. Prove that if $\{x_k\}$ and $\{y_k\}$ are sequences of real numbers such that
$$\sum_{k=1}^{\infty} x_k^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} y_k^2 < \infty, \quad \text{then} \quad \sum_{k=1}^{\infty} |x_k y_k| < \infty.$$
Hint: What can you say about the corresponding finite sums?
13. Find a non-zero vector in $\mathbb{R}^3$ which is orthogonal to both $(1, 0, 2)$ and $(3, -1, 1)$.
14. Prove that if u and v are vectors in an inner product space and $u \perp v$, then $||u + v||^2 = ||u||^2 + ||v||^2$.

7.2. Convergent Sequences of Vectors

In this section we study convergence of sequences of vectors in $\mathbb{R}^d$. The definitions and theorems in this topic are very similar to those of Chapter 2 on sequences of numbers.

Metric Spaces. As long as we are working in a space with a reasonable notion of distance between points, we can define and study convergent sequences and continuous functions. Such a space is called a *metric space.* The precise conditions for a space to be a metric space are defined below.

Definition 7.2.1. Let X be a set and let δ be a function which assigns to each pair (x, y) of elements of X a non-negative real number $\delta(x, y)$. Then δ is called a *metric* on X if, for all $x, y, z \in X$, the following conditions hold:

(a) $\delta(x, y) = \delta(y, x)$;

(b) $\delta(x, y) = 0$ if and only if $x = y$; and

(c) $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$.

A set X together with a metric δ on X is called a *metric space.* The number $\delta(x, y)$ is the distance between x and y in this metric space.

Conditions (a) and (b) above are called the symmetry and identity conditions, while condition (c) is the triangle inequality for metric spaces.

We will show that $\mathbb{R}^d$ is a metric space, as is any normed vector space.

Theorem 7.2.2. *If X is a normed vector space, then X is a metric space if its metric δ is defined by*
$$\delta(x, y) = ||x - y||.$$
In particular, $\mathbb{R}^d$ is a metric space in the Euclidean norm, as is $C(I)$ in the sup norm.

Proof. Parts (a), (b), and (c) of Theorem 7.1.10 are satisfied by the norm in any normed vector space. Part (b) with $a = -1$ implies that $||x - y|| = ||y - x||$ and so δ is symmetric. Part (c) implies that $||x - y|| = 0$, if and only if $x = y$, and so δ satisfies the identity condition. Part (a) implies (7.1.2), which shows that δ satisfies the triangle inequality. Thus, δ is a metric on X. $\square$

Example 7.2.17. Let the interval $(0,1)$ be considered a metric space with the usual distance between points as metric. Show that this is not a complete metric space.

Solution: The sequence $\{1/n\}$ is a Cauchy sequence since it converges in $\mathbb{R}$ to the point 0. However, since $0 \notin (0,1)$, this sequence does not converge in the metric space $(0,1)$. Hence, $(0,1)$ is not a complete metric space.

Exercise Set 7.2

1. Using only the definition of the limit of a sequence in $\mathbb{R}^d$ prove that
$$\lim \left(\frac{n}{1+n}, \frac{1-n}{n} \right) = (1,-1).$$

In each of the next four problems, decide if the sequence $\{x_n\}$ converges and find its limit if it does. Use limit theorems to justify your answers.

2. $x_n = \left(\frac{n^2+n-1}{3n^2+2}, \frac{n-1}{n+1} \right)$.
3. $x_n = (1+(-1)^n, 1/n, 1+1/n)$.
4. $x_n = (2^{-n}\sin(n\pi/4), 2^{-n}\cos(n\pi/4))$.
5. $x_n = (\ln(n+1) - \ln n, \sin(1/n))$.
6. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $\mathbb{R}^d$. Prove that if $\lim x_n = 0$ and if $\{y_n\}$ is bounded, then $\lim x_n \cdot y_n = 0$.
7. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}^d$ and let a_n be a bounded sequence of scalars. Prove that if either sequence has limit 0, then so does the sequence $\{a_n x_n\}$.
8. Prove that every convergent sequence in $\mathbb{R}^d$ is bounded.
9. If $x_n = (\sin n, \cos n, 1+(-1)^n)$, does the sequence $\{x_n\}$ in $\mathbb{R}^3$ have a convergent subsequence? Justify your answer.
10. Prove part (c) of Theorem 7.2.12.
11. If $x_n = (1/n, \sin(\pi n/2))$, find three convergent subsequences of $\{x_n\}$ which converge to three different limits.
12. If, for $x, y \in \mathbb{R}$, we set $\delta(x,y) = 0$ if $x = y$ and $\delta(x,y) = 1$ if $x \neq y$, prove that the result is a metric on $\mathbb{R}$. Thus, $\mathbb{R}$ with this metric is a metric space – one that is quite different from $\mathbb{R}$ with the usual metric.
13. Which sequences converge in the metric space of the previous exercise?
14. Let a and b be points of $\mathbb{R}^2$ and let X be the set of all smooth parameterized curves joining a to b in $\mathbb{R}^2$, with parameter interval $[0,1]$. That is, X is the set of all continuously differentiable functions $\gamma : [0,1] \to \mathbb{R}^2$, with $\gamma(0) = a$ and $\gamma(1) = b$. Show that if
$$\delta(\gamma_1, \gamma_2) = \sup\{||\gamma_1(t) - \gamma_2(t)|| : t \in [0,1]\},$$
then δ is a metric on X.

15. Show that the metric space of the previous exercise is not complete.
16. Let S be the surface of a sphere in $\mathbb{R}^3$. For $x, y \in S$ let $\delta(x, y)$ be the length of the shortest path on S joining x to y. Show that this is a metric on S.
17. Imagine a large building with many rooms. Let X be the set of rooms in this building and let $\delta(x, y)$ be the length of the shortest path along the hallways and stairways of the building that leads from room x to room y. Show that δ is a metric on X.

7.3. Open and Closed Sets

The open ball $B_r(x_0)$ and closed ball $\overline{B}_r(x_0)$, centered at $x_0 \in \mathbb{R}^d$, with radius $r > 0$, are defined by

$$B_r(x_0) = \{x \in \mathbb{R}^d : ||x - x_0|| < r\} \quad \text{and} \quad \overline{B}_r(x_0) = \{x \in \mathbb{R}^d : ||x - x_0|| \leq r\}.$$

Of course, open and closed balls centered at a given point and with a given radius may be defined in any metric space – one simply uses the metric distance $\delta(x, x_0)$ in place of the distance $||x - x_0||$ defined by the norm in $\mathbb{R}^d$.

Open intervals and closed intervals on the real line play an important part in the calculus of one variable. Open and closed balls are the direct analogues in $\mathbb{R}^d$ of open and closed intervals on the line. However, the geometry of $\mathbb{R}^d$ is much more complicated than that of the line. We will need the concepts of open and closed for sets that are far more complicated than balls. This leads to the following definition.

Definition 7.3.1. If U is a subset of $\mathbb{R}^d$, we will say that U is *open* if, for each point $x \in U$, there is an open ball centered at x which is contained in U. We will say that a subset of $\mathbb{R}^d$ is *closed* if its complement is open. A *neighborhood* of a point $x \in \mathbb{R}^d$ is any open set which contains x.

It might seem obvious that open balls are open sets and closed balls are closed sets. However, that is only because we have chosen to call them *open* balls and *closed* balls. We actually have to prove that they satisfy the conditions of the preceding definition. We do this in the next theorem.

Theorem 7.3.2. *In* $\mathbb{R}^d$,

(a) *the empty set* $\emptyset$ *is both open and closed;*

(b) *the whole space* $\mathbb{R}^d$ *is both open and closed;*

(c) *each open ball is open;*

(d) *each closed ball is closed.*

Proof. The empty set $\emptyset$ is open because it has no points, and so the condition that a set be open, stated in Definition 7.3.1, is vacuously satisfied. The set $\mathbb{R}^d$ is open because it contains any open ball centered at any of its points. Thus, $\emptyset$ and $\mathbb{R}^d$ are both open. Since they are complements of one another, they are also both closed.

with $\epsilon > 0$. This means that for each $\epsilon > 0$ there is an N such that $||x - x_n|| < \epsilon$ whenever $n \geq N$. That is, $\lim x_n = x$.

Conversely, if $\lim x_n = x$ and U is any neighborhood of x, we may choose an $\epsilon > 0$ such that the ball $B_\epsilon(x)$ is contained in U. By the definition of limit, for this ϵ there is an N such that $||x - x_n|| < \epsilon$ whenever $n \geq N$. Then $x_n \in B_\epsilon(x) \subset U$ whenever $n \geq N$. This completes the proof. □

Theorem 7.3.10. *If A is a subset of $\mathbb{R}^d$, then $\overline{A}$ is the set of all limits of convergent sequences in A. The set A is closed if and only if every covergent sequence in A converges to a point of A.*

Proof. If $x \in \overline{A}$, then each neighborhood of x contains a point of A by Theorem 7.3.7(b). In particular, each neighborhood of the form $B_{1/n}(x)$, for $n \in \mathbb{N}$, contains a point of A. We choose one and call it x_n. Since $||x - x_n|| < 1/n$, the sequence $\{x_n\}$ converges to x. Thus, each point in the closure of A is the limit of a sequence in A.

Conversely, suppose $x = \lim x_n$ for some sequence $\{x_n\}$ in A. By the previous theorem, each neighborhood of x contains points in this sequence. In particular, each neighborhood of x contains a point of A. Hence, $x \in \overline{A}$ by Theorem 7.3.7(b).

Since a set is closed if and only if it is its own closure, it follows that A is closed if and only if it contains all limits of convergent sequences in A. □

Exercise Set 7.3

1. Prove that the set $\{(x, y) \in R^2 : y > 0\}$ is an open subset of R^2.
2. Prove that every finite subset of $\mathbb{R}^d$ is closed.
3. Find the interior, closure, and boundary for the set
$$\{(x, y) \in \mathbb{R}^2 : 0 \leq x < 2,\ 0 \leq y < 1\}.$$
4. Find the interior, closure, and boundary for the set
$$\{(x, y) \in \mathbb{R}^2 : ||(x, y)|| < 1\} \cup \{(x, y) \in \mathbb{R}^2 : y = 0,\ -2 < x < 2\}.$$
5. Prove (c) and (d) of Theorem 7.3.3
6. Let A be an open set and B a closed set. If $B \subset A$, prove that $A \setminus B$ is open. If $A \subset B$, prove that $B \setminus A$ is closed.
7. Prove Theorem 7.3.7.
8. If E is a subset of $\mathbb{R}^d$, is the interior of the closure of E necessarily the same as the interior of E? Justify your answer.
9. If A and B are subsets of $\mathbb{R}^d$, show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Is the analogous statement true for $A \cap B$? Justify your answer.
10. If A and B are subsets of $\mathbb{R}^d$, prove that $(A \cap B)^\circ = A^\circ \cap B^\circ$. Is the analogous statement true for $A \cup B$? Justify your answer.
11. Let $\{x_n\}$ be a convergent sequence in $\mathbb{R}^d$ with limit x. Set
$$A = \{x_1, x_2, x_3, \ldots\} \cup \{x\};$$

that is, A is the set consisting of all the points occurring in the sequence together with the limit x. Show that A is a closed set.

12. Let $\{x_n\}$ be any sequence in $\mathbb{R}^d$ and let A be the set consisting of the points that occur in this sequence. Prove that the closure of A consists of A together with all limits of convergent subsequences of A.
13. Show that Theorem 7.3.10 remains true if $\mathbb{R}^d$ is replaced by any metric space.
14. Find the interior and closure of the set Q of rationals in $\mathbb{R}$.
15. If E is a subset of $\mathbb{R}^d$, show that $(\overline{E})^c = (E^c)^\circ$.

7.4. Compact Sets

In this section and the next, we study two topological properties, compactness and connectedness, that a subset of $\mathbb{R}^d$ may or may not have. A topological property of a set E is one that can be described using only knowledge of the open sets of $\mathbb{R}^d$ and their relationship to E. Thus, they are properties that can be defined in any toplological space. Compactness and connectedness are two such properties.

Open Covers. An open cover of a set $E \subset \mathbb{R}^d$ is a collection of open sets whose union contains E. An open cover of a set E may or may not have a finite subcover – that is, there may or may not be finitely many sets in the collection which also form a cover of E.

Example 7.4.1. The collection $\mathcal{U}$ of all open intervals of length 1/2 and with rational endpoints is clearly an open cover of the interval $[0,1]$. Show that it has a finite subcover.

Solution: The three intervals $(-1/8, 3/8)$, $(1/4, 3/4)$, and $(5/8, 9/8)$ belong to $\mathcal{U}$ and they cover $[0,1]$.

Example 7.4.2. The collection $\{(1/n, 1) : n = 1, 2, \dots\}$ is a collection of open sets which covers $(0,1)$. Does it have a finite subcover?

Solution: No. Since this collection of intervals is nested upward, any finite subcollection has a largest interval $(1/m, 1)$. Then the union of the sets in the subcollection is just $(1/m, 1)$ and this does not contain $(0,1)$.

Compactness. The above discussion leads to the following definition:

Definition 7.4.3. A subset K of $\mathbb{R}^d$ is called *compact* if every open cover of K has a finite subcover.

Note that Example 7.4.2 shows that the open interval $(0,1)$ is not compact, since it has an open cover with no finite subcover.

A subset E of $\mathbb{R}^d$ is bounded if there is a number R such that $||x|| \le R$ for every $x \in E$ – that is, if $E \subset \overline{B}_R(0)$ for some R.

Theorem 7.4.4. *Every compact subset K of $\mathbb{R}^d$ is bounded.*