Real Analysis
Math 127A-A, Winter 2019
Final: Solutions

1. [15pts] Let $A \subset \mathbb{R}$.
(a) Define what it means for $x \in \mathbb{R}$ to be an isolated point of A.
(b) Define what it means for $x \in \mathbb{R}$ to be a limit point of A.
(c) Define the closure $\bar{A}$ of A.

Solution

- (a) A real number $x \in \mathbb{R}$ is an isolated point of A if $x \in A$ and there exists $\epsilon > 0$ such that x is the only point of A in the interval $(x-\epsilon, x+\epsilon)$.
- (b) A real number $x \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$ if for every $\epsilon > 0$ there exists $a \in A$ with $a \neq x$ such that $a \in (x-\epsilon, x+\epsilon)$. Equivalently, $x \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$ if there is a sequence (a_n) of points $a_n \in A$ with $a_n \neq x$ such that $a_n \to x$ as $n \to \infty$.
- (c) The following are equivalent definitions: (i) $\bar{A} = A \cup L$ where L is the set of limit points of A; (ii) $\bar{A}$ is the set of limits of all convergent sequences in A; (iii) $\bar{A}$ is the intersection of all closed sets that contain A.

2. [20pts] Say if the following statements are true or false. If true, give a brief explanation (a complete proof is not required); if false, give a counterexample.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $I = (0, 1)$, then $f(I)$ is open.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $I = [0, 1]$, then $f(I)$ is closed.

(c) If (x_n) is a Cauchy sequence of real numbers, then $\{x_n : n \in \mathbb{N}\}$ is compact.

(d) If $A \subset \mathbb{R}$ is compact and $B \subset \mathbb{R}$ is closed, then $A \cap B$ is compact.

Solution

- (a) False. For example, if $f(x) = (x - 1/2)^2$, then $f((0, 1)) = [0, 1/4)$ isn't open.

- (b) True. The interval $[0, 1]$ is compact, so its continuous image is compact and therefore closed.

- (c) False. For example, the sequence $(1/n)$ converges so it is Cauchy, but its limit 0 doesn't belong to the set $A = \{1/n : n \in \mathbb{N}\}$, so A isn't closed.

- (d) True. Since A is compact it's closed and bounded. Then $A \cap B$ is closed and bounded, since the intersection of closed sets is closed and $A \cap B \subset A$ is bounded, so $A \cap B$ is compact.

3. [10pts] Prove that the polynomial equation $x^5 - 3x + 1 = 0$ has a root in the interval $0 < x < 1$.

Solution

- The polynomial function $p(x) = x^5 - 3x + 1$ is continuous on $[0, 1]$ and $p(0) = 1 > 0$, $p(1) = -1 < 0$, so the intermediate value theorem implies that there exists $0 < c < 1$ such that $p(c) = 0$.

4. [20pts] Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Justify all your steps. HINT. You can assume that $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

Solution

- Let

  $$S_n = \sum_{k=1}^{n} \frac{1}{(2k-1)^2}, \qquad T_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

  Then

  $$\begin{aligned} S_n &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} \\ &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots + \frac{1}{(2n-1)^2} + \frac{1}{(2n)^2} \\ &\quad - \left[\frac{1}{2^2} + \frac{1}{4^2} + \cdots + \frac{1}{(2n)^2}\right] \\ &= T_{2n} - \frac{1}{4}T_n. \end{aligned}$$

- From the hint, we have $T_n \to \pi^2/6$ and $T_{2n} \to \pi^2/6$ as $n \to \infty$. Taking the limit of the previous equation as $n \to \infty$, we get that

  $$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \lim_{n\to\infty} S_n = \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

5. [20pts] (a) Define the supremum $\sup A$ of a set $A \subset \mathbb{R}$.
(b) Define the sum of sets $A, B \subset \mathbb{R}$ by $A + B = \{a + b : a \in A,\ b \in B\}$. If A, B are nonempty sets that are bounded from above, prove that

$$\sup(A + B) = \sup A + \sup B.$$

Solution

- (a) $M = \sup A$ is the least upper bound of A. That is, $x \leq M$ for every $x \in A$, and for any $\epsilon > 0$ there exists $x \in A$ such that $x > M - \epsilon$.
- (b) Let $L = \sup A$ and $M = \sup B$ (which exist by the Dedekind completeness axiom for $\mathbb{R}$).
- If $x = a + b \in A + B$, then $x \leq L + M$, since L is an upper bound of A and M is an upper bound of B, so $L + M$ is an upper bound of $A + B$.
- Given any $\epsilon > 0$, there exists $a \in A$ such that $a > L - \epsilon/2$ and $b \in B$ such that $b > M - \epsilon/2$. It follows that $a + b > L + M - \epsilon$, which shows that $L + M$ is the least upper bound of $A + B$ and proves the result.

6. [20pts] (a) State the density property of the rational numbers $\mathbb{Q}$ in the real numbers $\mathbb{R}$.

(b) Let the sequence (r_n) be an enumeration of the rational numbers in $(0, 1)$, meaning that there is a one-to-one, onto function $f : \mathbb{N} \to \mathbb{Q} \cap (0, 1)$ such that $r_n = f(n)$. Prove that $\liminf_{n\to\infty} r_n = 0$ and $\limsup_{n\to\infty} r_n = 1$.

Solution

- (a) For every $x, y \in \mathbb{R}$ with $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$.
- (b) Consider
$$\limsup_{n\to\infty} r_n = \lim_{n\to\infty} \sup\{r_k : k \geq n\}.$$
Since $r_n < 1$ for every $n \in \mathbb{N}$, we have $\sup\{r_k : k \geq n\} \leq 1$. If $M < 1$, then repeated application of the density property implies that there exist infinitely many rational numbers $r \in \mathbb{Q}$ such that $M < r < 1$. It follows that for every $n \in \mathbb{N}$ there exists $k \geq n$ such that $M < r_k < 1$, so $\sup\{r_k : k \geq n\} = 1$, and $\limsup_{n\to\infty} r_n = 1$.
- A similar argument shows that $\inf\{r_k : k \geq n\} = 0$ for every $n \in \mathbb{N}$, and $\liminf_{n\to\infty} r_n = 0$.

7. [25pts] (a) Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$. State the ϵ-δ definition of $\lim_{x\to c} f(x) = L$.

(b) Prove from the ϵ-δ definition that $\lim_{x\to c} f(x) = L$ if and only for every sequence (x_n) in A such that $x_n \neq c$ and $x_n \to c$, one has $f(x_n) \to L$.

Solution

- (a) $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in A$ with $0 < |x - c| < \delta$.

- First, assume that $\lim_{x\to c} f(x) = L$, and suppose that (x_n) is a sequence in A such that $x_n \neq c$ and $x_n \to c$. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in A$ with $0 < |x - c| < \delta$. Since $x_n \to c$, there exists $N \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n > N$, so $|f(x_n) - L| < \epsilon$ for all $n > N$, which proves that $f(x_n) \to L$.

- To show that the sequential condition implies the limit, we prove the contrapositive statement.

- Assume that $f(x)$ doesn't converge to L as $x \to c$. Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there exists $x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon_0$. Taking $\delta = 1/n$ for $n \in \mathbb{N}$, we find that there exists $x_n \in A$ with $0 < |x_n - c| < 1/n$ and $|f(x_n) - L| \geq \epsilon_0$. It follows that (x_n) is a sequence in A with $x_n \neq c$ and $x_n \to c$, but $(f(x_n))$ doesn't converge to L, so the sequential condition doesn't hold.