Real Analysis

Math 127A-B, Winter 2019

Final: Solutions

1. [15pts] Give precise and complete statements of the following theorems.

(a) The alternating series test.

(b) The extreme value theorem.

(c) The intermediate value theorem.

Solution

- (a) If (a_n) is a decreasing sequence of positive numbers $a_n \geq 0$ such that $a_n \to 0$, then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.
- (b) If $f : K \to \mathbb{R}$ is continuous on a compact set K, then f is bounded and attains its maximum and minimum values.
- (c) If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there exists $a < c < b$ such that $f(c) = L$.

2. [20pts] Say if the following statements are true or false. If true, give a brief explanation (a complete proof is not required); if false, give a counterexample.

(a) If (x_n) is a bounded sequence of real numbers and $\limsup x_n \leq \liminf x_n$, then (x_n) converges.

(b) If a set $A \subset \mathbb{R}$ has the property that for every $\epsilon > 0$ there exists $x, y \in A$ such that $0 < |x - y| < \epsilon$, then A has a limit point.

(c) If $A \subset \mathbb{R}$ is bounded, then the closure of A is compact.

(d) If $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ is a nested sequence of nonempty compact sets $K_n \subset \mathbb{R}$ and $A = \cap_{n=1}^{\infty} K_n$, then $\sup K_n \to \sup A$ as $n \to \infty$.

Solution

- (a) True. We always have $\limsup x_n \geq \liminf x_n$, so $\limsup x_n = \liminf x_n$, which implies that (x_n) converges.

- (b) False. For example, $A = \mathbb{N} \cup \{n + 1/n : n \in \mathbb{N}\}$ satisfies the given condition, but every point in A is an isolated point of A.

- (c) True. If A is bounded, then $\bar{A}$ is bounded, and $\bar{A}$ is always closed, so $\bar{A}$ is compact.

- (d) True. Since the K_n and A are compact and nonempty, they contain their suprema. If $b_n = \max K_n$, then (b_n) is bounded and decreasing, so $b_n \to b \in \mathbb{R}$. Since K_n is closed, we have $b \in K_n$ for every $n \in \mathbb{N}$, so $b \in A$, which implies that $b \leq \max A$. Also, since $A \subset K_n$, we have $\max A \leq b_n$, so $\max A \leq b$, meaning that $b = \max A$ and $\max K_n \to \max A$.

- As a special case, if the $K_n = [a_n, b_n]$ are nested compact intervals, then $A = [a, b]$ where $a_n \uparrow a$ and $b_n \downarrow b$.

3. [15pts] (a) Suppose that $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$. State the ϵ-δ definition of $\lim_{x\to c} f(x) = L$.

(b) Let $A = [0, \infty) \setminus \{9\}$, and define $f : A \to \mathbb{R}$ by

$$f(x) = \frac{x-9}{\sqrt{x}-3}.$$

Prove from the ϵ-δ definition that $\lim_{x\to 9} f(x) = 6$. Write your proof starting with the statement: "Let $\epsilon > 0$ be given."

Solution

- (a) $\lim_{x\to c} f(x) = L$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in A$ with $0 < |x - c| < \delta$.

- (b) Let $\epsilon > 0$ be given. Choose $\delta = 3\epsilon$. Then $0 < |x - 9| < \delta$ implies that

$$\begin{aligned}
|f(x) - 6| &= \left|\frac{x-9}{\sqrt{x}-3} - 6\right| \\
&= \left|\sqrt{x} + 3 - 6\right| \\
&= \left|\sqrt{x} - 3\right| \\
&= \left|\frac{x-9}{\sqrt{x}+3}\right| \\
&\leq \frac{1}{3}|x - 9| \\
&< \frac{1}{3}\delta \\
&< \epsilon,
\end{aligned}$$

which proves the result.

4. [20pts] Let $0 < a < 1$ and define $x_n = na^n$ for $n \in \mathbb{N}$.
(a) Prove that there exists $N \in \mathbb{N}$ and $0 < K < 1$ such that $x_{n+1} \leq Kx_n$ for all $n \geq N$.
(b) Show that $\lim_{n\to\infty} na^n = 0$.

Solution

- (a) We have

$$\frac{x_{n+1}}{x_n} = \left(1 + \frac{1}{n}\right) a \to a \qquad \text{as } n \to \infty.$$

  Taking $\epsilon = (1 - a)/2 > 0$ in the ϵ-N definition of the limit, we get that there exists $N \in \mathbb{N}$ such that $x_{n+1}/x_n < K$ for all $n \geq N$ where $K = (1 + a)/2 < 1$.

- (b) It follows from (a) that $0 \leq x_{N+r} \leq K^r x_N$ for all $r \geq 0$. Since $K^r \to 0$ as $r \to \infty$ for $0 < K < 1$, the squeeze theorem implies that $x_n \to 0$ as $n \to \infty$.

5. [20pts] (a) Prove by induction that for every $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}. \tag{1}$$

(b) Deduce that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = 2.$$

You can assume the result of Problem 4(b).

Solution

- (a) One verifies immediately that (1) holds for $n = 1$. Suppose that (1) holds for some $n \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{k=1}^{n+1} \frac{k}{2^k} &= \sum_{k=1}^{n} \frac{k}{2^k} + \frac{n+1}{2^{n+1}} \\ &= 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} \\ &= 2 - \frac{(n+1)+2}{2^{n+1}}, \end{aligned}$$

  so (1) holds for $n+1$, and the result follows for every $n \in \mathbb{N}$ by induction.

- (b) Since $1/2^n \to 0$ and $n/2^n \to 0$, we get that

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{2^k} = \lim_{n\to\infty} \left(2 - \frac{n+2}{2^n}\right) = 2.$$

Remark. A geometrical proof for the sum of this series was given by the medieval philosopher and mathematician Nicole Oresme around 1350.

6. [20pts] (a) State the Bolzano-Weierstrass theorem.

(b) Suppose that (x_n) is a bounded sequence of real numbers with the property that every convergent subsequence converges to the same limit $L \in \mathbb{R}$. Prove that (x_n) converges to L. HINT. Assume that (x_n) doesn't converge to L and derive a contradiction.

(c) Can a divergent sequence have the property that every convergent subsequence converges to the same limit?

Solution

- (a) Every bounded sequence of real numbers has a convergent subsequence.

- (b) Suppose for contradiction that (x_n) does not converge to L. Then there exists $\epsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there exists $n > N$ with $|x_n - L| \geq \epsilon_0$. It follows that we can choose a subsequence (x_{n_k}) with $n_1 < n_2 < \cdots < n_k < \dots$ such that $|x_{n_k} - L| \geq \epsilon_0$ for every $k \in \mathbb{N}$. This subsequence is bounded, since (x_n) is bounded, so by the Bolzano-Weierstrass theorem, we can extract a convergent subsequence of (x_{n_k}). This subsequence is a convergent subsequence of (x_n), but it cannot converge to L, contradicting our assumption. It follows that (x_n) must converge to L.

- (c) Yes. For example, the divergent sequence (x_n) with $x_n = n$ has no convergent subsequences, so it satisfies this condition vacuously. Alternatively, the only convergent subsequences of the sequence (x_n) with

  $$x_n = \begin{cases} n & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

  are the ones whose terms are eventually equal to 1, so all convergent subsequences have the same limit.

7. [20pts] (a) Define what it means for $f : A \to \mathbb{R}$ to be uniformly continuous on $A \subset \mathbb{R}$.

(b) Suppose that $f : A \to \mathbb{R}$ is uniformly continuous on A. Let (x_n) be a convergent sequence in A, whose limit need not belong to A, and define $y_n = f(x_n)$ for $n \in \mathbb{N}$. Prove that the sequence (y_n) converges.

(c) Does the result in (b) remain true if $f : A \to \mathbb{R}$ is only assumed to be continuous on A? Justify your answer.

Solution

- (a) $f : A \to \mathbb{R}$ is uniformly continuous on A if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for every $x, y \in A$ with $|x - y| < \delta$.

- (b) Let $\epsilon > 0$. Choose $\delta > 0$ such that $x, y \in A$ and $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$. Since (x_n) converges, it is Cauchy, so there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \delta$ for all $m, n > N$. It follows that $|f(x_m) - f(x_n)| < \epsilon$ for all $m, n > N$, which proves that (y_n) is Cauchy, so it converges.

- (c) No. For example, define the continuous function $f : (0, 1) \to \mathbb{R}$ by $f(x) = 1/x$ and let $x_n = 1/n$. Then $x_n \to 0$ but $f(x_n) = n$ diverges.

Remark. A similar argument shows that if $f : A \to \mathbb{R}$ is uniformly continuous and (x_n), (x'_n) are any sequences in A that converge to c, then $(f(x_n))$, $(f(x'_n))$ converge to the same limit. It follows that a uniformly continuous function $f : A \to \mathbb{R}$ has a unique extension to a continuous (in fact, uniformly continuous) function $\bar{f} : \bar{A} \to \mathbb{R}$ on the closure of the domain of f, by defining $\bar{f}(c) = \lim_{n\to\infty} f(x_n)$ for $x_n \to c$ at every limit point c of A. As the example in (c) shows, such an extension need not exist if $f : A \to \mathbb{R}$ is only continuous on A.