The lim inf and lim sup and Cauchy sequences

1 The lim sup and lim inf

We begin by stating explicitly some immediate properties of the sup and inf, which we use below.

Proposition 1. (a) If $A \subset \mathbb{R}$ is a nonempty set, then $\inf A \leq \sup A$. (b) If $A \subset B$, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.

Proof. (a) If $x \in A$, then $\inf A \leq x \leq \sup A$, so the result follows. (b) If $A \subset B$, then $\sup B$ is an upper bound of A, so $\sup A \leq \sup B$. Similarly, $\inf B$ is a lower bound of A, so $\inf A \geq \inf B$. □

Suppose that (x_n) is a bounded sequence, meaning that there exist $m, M \in \mathbb{R}$ such that

$$m \leq x_n \leq M \qquad \text{for all } n \in \mathbb{N}.$$

Let $T_n \subset \mathbb{R}$ be the set of terms of the tail of the sequence starting at x_n,

$$T_n = \{x_k : k \geq n\}.$$

Then T_n is bounded from above by M and bounded from below m, so

$$y_n = \sup T_n, \qquad z_n = \inf T_n$$

exist, and

$$m \leq z_n \leq y_n \leq M. \tag{1}$$

Moreover, $T_{n+1} \subset T_n$, so $y_{n+1} \leq y_n$ and $z_{n+1} \geq z_n$. It follows that (y_n) is a decreasing sequence that is bounded from below by m, and (z_n) is an increasing sequence that is bounded from above by M, so both sequences converge. Their limits define the lim sup and lim inf of the original sequence.

Definition 2. Let (x_n) be a bounded squence. Then

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left[\sup\{x_k : k \geq n\}\right], \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left[\inf\{x_k : k \geq n\}\right].$$

That is,

$$\limsup x_n = \lim y_n, \qquad \liminf x_n = \lim z_n.$$

It follows from (1) and the order properties of limits that

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n. \tag{2}$$

Theorem 3. A sequence (x_n) converges to $x \in \mathbb{R}$ if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x.$$

Proof. Suppose that the lim sup and lim inf of (x_n) are both equal to $x \in \mathbb{R}$. Then $y_n \to x$ and $z_n \to x$. The definition of y_n and z_n implies that $z_n \le x_n \le y_n$ for every $n \in \mathbb{N}$, so the "squeeze" theorem implies that $x_n \to x$.

Conversely, suppose that $x_n \to x$. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$x - \epsilon < x_n < x + \epsilon \qquad \text{for every } n > N.$$

It follows that

$$x - \epsilon \le \inf\{x_k : k \ge n\} \le \sup\{x_k : k \ge n\} \le x + \epsilon \qquad \text{for every } n > N,$$

which shows that

$$|y_n - x| \le \epsilon, \quad |z_n - x| \le \epsilon \qquad \text{for every } n > N.$$

Hence, $y_n \to x$ and $z_n \to x$, so $\limsup x_n = \liminf x_n = x$. □

2 Cauchy sequences

A Cauchy sequence is a sequence whose terms eventually get arbitrarily close together.

Definition 4. A sequence (x_n) of real numbers is a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_m - x_n| < \epsilon \qquad \text{for all } m, n > N.$$

Every convergent sequence is Cauchy, and the completeness of $\mathbb{R}$ implies that every Cauchy sequence converges.

Theorem 5. A sequence of real numbers converges if and only if it is a Cauchy sequence.

Proof. First suppose that (x_n) converges to a limit $x \in \mathbb{R}$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2} \qquad \text{for all } n > N.$$

It follows that if $m, n > N$, then

$$|x_m - x_n| \le |x_m - x| + |x - x_n| < \epsilon,$$

which implies that (x_n) is Cauchy. (This direction doesn't use the completeness of $\mathbb{R}$; for example, it holds equally well for sequence of rational numbers that converge in $\mathbb{Q}$.)

Conversely, suppose that (x_n) is Cauchy. Then there is $N_1 \in \mathbb{N}$ such that

$$|x_m - x_n| < 1 \qquad \text{for all } m, n > N_1.$$

It follows that if $n > N_1$, then

$$|x_n| \le |x_n - x_{N_1+1}| + |x_{N_1+1}| \le 1 + |x_{N_1+1}|.$$

Hence the sequence is bounded with

$$|x_n| \le \max\{|x_1|, |x_2|, \dots, |x_{N_1}|, 1 + |x_{N_1+1}|\}.$$

Since the sequence is bounded, its lim sup and lim inf exist. We claim they are equal. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that the Cauchy condition in Definition 4 holds. Then

$$x_n - \epsilon < x_m < x_n + \epsilon \qquad \text{for all } m \ge n > N.$$

It follows that for all $n > N$ we have

$$x_n - \epsilon \le \inf\{x_m : m \ge n\}, \quad \sup\{x_m : m \ge n\} \le x_n + \epsilon,$$

which implies that

$$\sup\{x_m : m \ge n\} - \epsilon \le \inf\{x_m : m \ge n\} + \epsilon.$$

Taking the limit as $n \to \infty$, we get that

$$\limsup_{n\to\infty} x_n - \epsilon \le \liminf_{n\to\infty} x_n + \epsilon,$$

and since $\epsilon > 0$ is arbitrary, we have

$$\limsup_{n\to\infty} x_n \le \liminf_{n\to\infty} x_n.$$

In view of (2), it follows that $\limsup x_n = \liminf x_n$, so the sequence converges by Theorem 3. □