Solution to 1.3.3

1.3.3 (a) Let $A \subset \mathbb{R}$ be nonempty and bounded from below and $B \subset \mathbb{R}$ the set of lower bounds of A. Show that $\sup B = \inf A$.

(b) Explain why we don't need to assert the existence of greatest lower bounds in the Dedekind completeness axiom for $\mathbb{R}$.

Solution

- (a) We proceed in three steps: (i) prove that $\sup B$ exists; (ii) prove that $\sup B$ is a lower bound of A; (iii) prove that $\sup B$ is the greatest lower bound of A.

- (i) First, $B \neq \varnothing$ since A is bounded from below. Second, if $a \in A$, then $b \leq a$ for every $b \in B$, since b is a lower bound of A, so B is bounded from above by a. In particular, B is bounded from above since $A \neq \varnothing$. Since B is nonempty and bounded from above, its supremum $\sup B$ exists by the Dedekind completeness axiom for $\mathbb{R}$.

- (ii) Since every $a \in A$ is an upper bound of B, we have $\sup B \leq a$ since $\sup B$ is the *least* upper bound of B. Hence $\sup B$ is a lower bound of A. (So $\sup B \in B$ and $\sup B$ is the maximal element of B.)

- (iii) If b is any lower bound of A, then $b \in B$, so $b \leq \sup B$ since $\sup B$ is an upper bound of B. It follows that $\sup B$ is the greatest lower bound of A, which proves that $\inf A$ exists and is equal to $\sup B$.

- (b) The preceding argument shows that we can deduce the existence of the infimum of a nonempty set that is bounded from below from the existence of the supremum of a nonempty set that is bounded from above, so it's not necessary to require the existence of infima in the completeness axiom. Note that we didn't assume anywhere in (a) that $\inf A$ exists; rather we showed that $\sup B$ exists and is the greatest lower bound of A.

Remark. In class, we deduced the existence of infima from suprema by noting that $\inf A = -\sup(-A)$. The proof in this question has the advantage that it only uses the order properties of $\mathbb{R}$, not the field properties, so it applies equally well to any Dedekind complete ordered set.