Real Analysis
Math 127A, Winter 2019
Midterm 1: Sample Problems

1. Say if the following statements are true or false. If true, give a brief explanation (a complete proof is not required); if false, give a counterexample.

(a) If $A, B \subset \mathbb{R}$ are nonempty sets that are bounded from above and $A \cap B$ is nonempty, then

$$\sup(A \cap B) = \min\{\sup A, \sup B\}.$$

(b) If $A, B \subset \mathbb{R}$ are nonempty sets that are bounded from above, then

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

(c) If (x_n) is a convergent sequence and (y_n) is a sequence such that $y_n \to \infty$ as $n \to \infty$, then there exists $N \in \mathbb{N}$ such that $y_n > x_n$ for every $n > N$.

(d) If a sequence is bounded from above, then it has a convergent subsequence.

2. Suppose that $x \geq 0$. Prove by induction that

$$1 + nx \leq (1+x)^n \qquad \text{for every } n \in \mathbb{N}.$$

3. (a) Let $A \subset \mathbb{R}$ be a nonempty set that is bounded from above. Prove that there is a sequence (a_n) of points $a_n \in A$ such that $a_n \to \sup A$ as $n \to \infty$.

(b) Can the sequence in (a) be chosen so that it is increasing? Briefly explain your answer (a complete proof is not required).

4. (a) State the definition of a Cauchy sequence (x_n) of real numbers.

(b) Let $x_n = \sqrt{n}$. Prove that for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|x_{n+1} - x_n| < \epsilon \qquad \text{for every } n > N.$$

(c) Is (x_n) a Cauchy sequence?

5. (a) Let (x_n) be a sequence of real numbers and $x \in \mathbb{R}$. Define $y_n = |x_n - x|$. Prove that the sequence (x_n) is bounded if and only if the sequence (y_n) is bounded.

(b) Prove that $\lim_{n\to\infty} x_n = x$ if and only if $\limsup_{n\to\infty} |x_n - x| = 0$.