Real Analysis

Math 127A, Winter 2019

Midterm 2: Sample Problems

1. Say if the following statements are true or false. If true, give a brief explanation (a complete proof is not required); if false, give a counterexample.

(a) If $\sum a_n$ diverges and $0 \le a_n \le b_n$, then $\sum b_n$ diverges.

(b) If the series $\sum a_n$ is conditionally convergent, then the series $\sum \sqrt{n} a_n$ diverges.

(c) If $A \subset \mathbb{R}$ has closure $\bar{A} = \mathbb{R}$, then A is uncountable.

(d) If $A \subset \mathbb{R}$ and every $a \in A$ is an isolated point of A, then A is closed.

2. (a) State the Cauchy condition for the convergence of a series $\sum_{n=1}^{\infty} a_n$.

(b) Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence of real numbers and $(a_{n_k})_{k=1}^{\infty}$ is a subsequence. If the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, prove that the series $\sum_{k=1}^{\infty} a_{n_k}$ converges.

(c) Does the result in (b) remain true if $\sum_{n=1}^{\infty} a_n$ converges conditionally? Justify your answer.

3. Determine the convergence of the following series and justify your answers (you can use any test):

$$\text{(a)} \sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n!}}; \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{n-1}{n^2+1}; \qquad \text{(c)} \sum_{n=1}^{\infty} \left(\frac{n-1}{n} - \frac{n}{n+1} \right).$$

4. (a) Define an open set $G \subset \mathbb{R}$.

(b) Let $A \subset \mathbb{R}$ and $\epsilon > 0$. Prove that the following set is open:

$$G = \{x \in \mathbb{R} : |x - a| < \epsilon \text{ for some } a \in A\}.$$

5. Suppose that $A \subset \mathbb{R}$ is nonempty and closed, and $b \in \mathbb{R}$. Prove that there exists $a \in A$ such that $|a - b| = \inf\{|x - b| : x \in A\}$. Is a unique?