Real Analysis
Math 127A-A, Winter 2019
Midterm 1: Solutions

1. [15pts] What are the axioms that define the real numbers $\mathbb{R}$? You should explain briefly what the axioms say, but you do not have to write them out in detail.

Solution

- The real numbers $\mathbb{R}$ are a Dedekind-complete ordered field.
- "Field" means that $\mathbb{R}$ is equipped with addition and multiplication operations $(+, \cdot)$ with identity elements $0, 1 \in \mathbb{R}$ that satisfy the usual properties: $(\mathbb{R}, +, 0)$ and $(\mathbb{R}^*, \cdot, 1)$ are commutative groups, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$; and multiplication is distributive over addition.
- "Ordered" means that $\mathbb{R}$ is linearly ordered by an order relation $<$ that is compatible with its algebraic operations.
- "Dedekind-complete" means that every nonempty subset of $\mathbb{R}$ that is bounded from above has a supremum.

2. [20pts] Say if the following statements are true or false. If true, give a brief explanation (a complete proof is not required); if false, give a counterexample.

(a) If $A \subset \mathbb{R}$ is nonempty and bounded from above, then $\sup A \in A$.

(b) If (x_n) and (y_n) are sequences such that $x_n \to 0$ as $n \to \infty$ and (y_n) is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

(c) If an increasing sequence (x_n) contains a convergent subsequence (x_{n_k}), then (x_n) converges.

(d) If $x_n = \sin n$, then the sequence (x_n) has a convergent subsequence.

Solution

- (a) False. For example, $\sup(0, 1) = 1 \notin (0, 1)$.
- (b) True. If $|y_n| \le M$, then $0 \le |x_n y_n| \le M|x_n|$, and $x_n y_n \to 0$ by the squeeze theorem.
- (c) True. Let $s = \sup\{x_{n_k} : k \in \mathbb{N}\}$. If $n \in \mathbb{N}$, then there exists $k \in \mathbb{N}$ such that $n \le n_k$. Since the sequence is increasing, we have $x_n \le x_{n_k} \le s$, so the sequence (x_n) is bounded from above by s, and it converges by the monotone convergence theorem (to the same limit as its subsequences).
- (d) True. We have $|\sin n| \le 1$ for every $n \in \mathbb{N}$, so the sequence is bounded, and it has a convergent subsequence by the Bolzano-Weierstrass theorem.

3. (a) State what it means for the rational numbers to be dense in the real numbers.

(b) Prove that every real number is the limit of a sequence of rational numbers.

Solution

- (a) For every $x, y \in \mathbb{R}$ with $x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$.
- (b) Let $x \in \mathbb{R}$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, for every $n \in \mathbb{N}$, there exists $r_n \in \mathbb{Q}$ such that
$$x - \frac{1}{n} < r_n < x + \frac{1}{n}.$$
Since $x - 1/n \to x$ and $x + 1/n \to x$ as $n \to \infty$, the squeeze theorem implies that $r_n \to x$ as $n \to \infty$.

4. [20pts] (a) State the definition of a Cauchy sequence (x_n) of real numbers.
(b) Suppose that a sequence (x_n) satisfies

$$|x_{n+1} - x_n| < \frac{1}{2^n} \qquad \text{for every } n \in \mathbb{N}. \tag{1}$$

Prove that (x_n) is a Cauchy sequence. HINT. Recall that

$$\sum_{k=0}^{N} \frac{1}{2^k} = \frac{1 - (1/2)^{N+1}}{1 - (1/2)} < 2. \tag{2}$$

Solution

- (a) A sequence (x_n) is a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon$ for all $m, n > N$.

- (b) Let $m, n \in \mathbb{N}$ with $m > n$. Then

$$\begin{aligned} x_m - x_n &= x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n \\ &= \sum_{k=n}^{m-1} (x_{k+1} - x_k) \end{aligned}$$

  Using the triangle inequality and (1)–(2), we get that

$$\begin{aligned} |x_m - x_n| &\le \sum_{k=n}^{m-1} |x_{k+1} - x_k| \\ &\le \sum_{k=n}^{m-1} \frac{1}{2^k} \\ &\le \frac{1}{2^n} \sum_{j=0}^{m-n-1} \frac{1}{2^j} \\ &< \frac{1}{2^{n-1}}. \end{aligned}$$

- Let $\epsilon > 0$. Since $1/2^{n-1} \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $1/2^{N-1} < \epsilon$. Then for every $m > n > N$, we have

$$|x_m - x_n| < \frac{1}{2^{n-1}} < \frac{1}{2^{N-1}} < \epsilon,$$

  which proves that (x_n) is Cauchy.