Real Analysis
Math 127A-A, Winter 2019
Midterm 2: Solutions

1. [15pts] Let $A \subset \mathbb{R}$.

(a) Define what it means for $x \in \mathbb{R}$ to be an isolated point of A.

(b) Define what it means for $x \in \mathbb{R}$ to be a limit point of A.

(c) Define the closure $\bar{A}$ of A.

Solution

- (a) A real number $x \in \mathbb{R}$ is an isolated point of A if $x \in A$ and there exists $\epsilon > 0$ such that x is the only point of A in the interval $(x-\epsilon, x+\epsilon)$.

- (b) A real number $x \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$ if for every $\epsilon > 0$ there exists $a \in A$ with $a \neq x$ such that $a \in (x-\epsilon, x+\epsilon)$. Equivalently, $x \in \mathbb{R}$ is a limit point of $A \subset \mathbb{R}$ if there is a sequence (a_n) of points $a_n \in A$ with $a_n \neq x$ such that $a_n \to x$ as $n \to \infty$.

- (c) The following are equivalent definitions: (i) $\bar{A} = A \cup L$ where L is the set of limit points of A; (ii) $\bar{A}$ is the set of limits of all convergent sequences in A; (iii) $\bar{A}$ is the intersection of all closed sets that contain A.

2. [20pts] Say if the following statements are true or false. If true, give a brief explanation (a complete proof is not required); if false, give a counterexample.

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{2^n + n^2}{n^2 2^n + 1} \right)$$

converges absolutely.

(c) A finite subset of $\mathbb{R}$ is closed.

(d) If $G_n \subset \mathbb{R}$ is open for every $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} G_n$ is open.

Solution

- (a) False. For example, the harmonic series $\sum 1/n$ diverges but $1/n \to 0$ as $n \to \infty$.

- (b) True. We have

  $$\left| (-1)^{n+1} \left(\frac{2^n + n^2}{n^2 2^n + 1} \right) \right| \leq \frac{1}{n^2} + \frac{1}{2^n},$$

  and

  $$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{2^n} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

  converges, so the series converges absolutely by the comparison test.

- (c) True. A singleton set $\{x\}$ is closed and the finite union of closed sets is closed.

- (d) False. For example, $G_n = (-\infty, 1/n)$ is open, but

  $$\bigcap_{n=1}^{\infty} G_n = (-\infty, 0]$$

  is not.

3. [20pts] Let (a_n) be a sequence, where $n \in \mathbb{N}$, and define a sequence (b_n) by $b_n = a_{2n-1} + a_{2n}$.

(a) If the series $\sum_{n=1}^{\infty} a_n$ converges, prove that the series

$$\sum_{n=1}^{\infty} b_n = (a_1 + a_2) + (a_3 + a_4) + (a_5 + a_6) + \dots$$

converges to the same sum.

(b) Does the convergence of $\sum_{n=1}^{\infty} b_n$ imply the convergence of $\sum_{n=1}^{\infty} a_n$?

Solution

- (a) Let $S_n = \sum_{k=1}^{n} a_k$ denote the nth partial sum of $\sum a_k$. Then

$$S_n \to S = \sum_{k=1}^{\infty} a_k \qquad \text{as } n \to \infty.$$

- Let $T_m = \sum_{k=1}^{m} b_k$ denote the mth partial sum of $\sum b_k$. Then $T_m = S_{2m}$, meaning that (T_m) is a subsequence of the convergent sequence (S_n), so (T_m) converges to the same limit S, and

$$\sum_{k=1}^{\infty} b_k = \lim_{m \to \infty} T_m = \sum_{k=1}^{\infty} a_k.$$

- (b) The convergence of $\sum b_n$ doesn't imply the convergence of $\sum a_n$. For example, if $a_n = (-1)^{n+1}$, then $b_n = 0$ for every $n \in \mathbb{N}$, and $\sum b_n$ converges, but the partial sums of $\sum a_n$ oscillate between 1 and 0, so the series diverges.

4. [20pts] (a) Define what it means for $A \subset \mathbb{R}$ to be an open set.
(b) Define the sum of sets $A, B \subset \mathbb{R}$ to be the set

$$A + B = \{a + b : a \in A,\, b \in B\}.$$

If A is open, prove that $A + B$ is open.

Solution

- (a) A set $A \subset \mathbb{R}$ is open if for every $x \in A$ there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset A$.

- (b) Let $x \in A + B$, so $x = a + b$ for some $a \in A$ and $b \in B$. Since A is open, there exists $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset A$. It follows that $(a - \epsilon + b, a + \epsilon + b) \subset A + B$, or $(x - \epsilon, x + \epsilon) \subset A + B$, which proves that $A + B$ is open.